

PARTIALLY DISSIPATIVE SYSTEMS IN THE CRITICAL REGULARITY SETTING, AND STRONG RELAXATION LIMIT

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ABSTRACT. Many physical phenomena may be modelled by first order hyperbolic equations with degenerate dissipative or diffusive terms. This is the case for example in gas dynamics, where the mass is conserved during the evolution, but the momentum balance includes a diffusion (viscosity) or damping (relaxation) term, or, in numerical simulations, of conservation laws by relaxation schemes.

Such so-called *partially dissipative systems* have been first pointed out by S.K. Godunov in a short note in Russian in 1961. Much later, in 1984, S. Kawashima highlighted in his PhD thesis a simple criterion ensuring the existence of global strong solutions in the vicinity of a linearly stable constant state. This criterion has been revisited in a number of research works. In particular, K. Beauchard and E. Zuazua proposed in 2010 an explicit method for constructing a Lyapunov functional allowing to refine Kawashima’s results and to establish global existence results in some situations that were not covered before.

These notes originate essentially from the PhD thesis of T. Crin-Barat that was initially motivated by an earlier observation of the author in a Chapter of the handbook coedited by Y. Giga and A. Novotný. Our main aim is to adapt the method of Beauchard and Zuazua to a class of symmetrizable quasilinear hyperbolic systems (containing the compressible Euler equations), in a critical regularity setting that allows to keep track of the dependence with respect to e.g. the relaxation parameter. Compared to Beauchard and Zuazua’s work, we exhibit a ‘damped mode’ that will have a key role in the construction of global solutions with critical regularity, in the proof of optimal time-decay estimates and, last but not least, in the study of the strong relaxation limit. For simplicity, we here focus on a simple class of partially dissipative systems, but the overall strategy is rather flexible, and adaptable to much more involved situations.

INTRODUCTION

An important recent mathematical literature has been devoted to the study of first order systems of conservation laws. These systems that come into play in the description of a number phenomena in mechanics, physics or engineering typically read

$$(1) \quad \partial_t f^0(V) + \sum_{k=1}^d \partial_{x_k}(f^k(V)) = 0$$

where the vector-fields f^k , $k = 0, \dots, d$ are defined on some open subset \mathcal{O} of \mathbb{R}^n , and the unknown V depends on the time variable $t \in \mathbb{R}_+ \triangleq [0, \infty)$ and on the space variable $x \in \mathbb{R}^d$.

Under rather general conditions, for example whenever (1) is Friedrichs-symmetrizable, it is well known that for any \bar{V} in \mathcal{O} and initial data $V_0 : \mathbb{R}^d \rightarrow \mathcal{O}$ such that $V_0 - \bar{V}$ belongs to some Sobolev space $H^s(\mathbb{R}^d)$ with $s > 1 + d/2$, then (1) supplemented with initial

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data V_0 admits a unique classical solution V on some time interval $[-T, T]$, satisfying $(V - \bar{V}) \in \mathcal{C}_b([-T, T]; H^s(\mathbb{R}^d))$ (the reader may find the detailed statement and the proof in e.g. [4, Chap. 10]). At the same time, for most systems of the above type, smooth solutions (even small ones) blow-up after finite time.

In many physical systems however, friction or diffusion phenomena (through e.g. thermal conduction or viscosity) cannot be neglected. Typically, they act on some components of the unknown, while other components remain unaffected. An informative example is gas dynamics where the mass is conserved (as well as the entropy in the isentropic case). In order to have an accurate description corresponding to these situations, it is thus suitable to add in (1) zero (friction) or second (diffusion) order terms that act on a part of the unknown but, possibly, not on all components. The resulting class of systems is named, depending on the authors and on the context, hyperbolic-parabolic, partially diffusive or partially dissipative. It has been extensively studied since the pioneering work by S. Kawashima in his PhD thesis [24]. One of the main issues is to find as weak as possible conditions ensuring the existence of global solutions close to constant states, to describe their long time asymptotics and, where applicable, to study the convergence to some limit system.

Rather than writing out now the class of systems that enter in our study, let us give a simple example from multi-dimensional gas dynamics. In the barotropic and isothermal case, the governing equations then read:

$$(2) \quad \begin{cases} \partial_t \varrho + \operatorname{div}_x(\varrho v) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t(\varrho v) + \operatorname{div}_x(\varrho v \otimes v) + \nabla_x P = A(\varrho, v) & \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \end{cases}$$

Above, $\varrho = \varrho(t, x) \in \mathbb{R}_+$ stands for the density of the gas, and $v = v(t, x) \in \mathbb{R}^d$, for the velocity. The pressure $P = P(\varrho)$ is a given function of the density. A typical example is the *isentropic pressure law* $P(\varrho) = a\varrho^\gamma$ with $a > 0$ and $\gamma > 1$. The first equation corresponds to the mass conservation and the second one, to the momentum balance. We assume that the fluid domain is the whole space which, somehow, means that boundary effects are neglected. This is a fundamental assumption for our analysis, that strongly relies on Fourier methods.

Regarding A , the usual assumptions are:

- either A is identically zero: then (2) is the *barotropic compressible Euler equations* that is known to be Friedrichs-symmetrizable (again, refer e.g. to [4, Chap. 10]) and thus enters in the class considered in (1);
- or $A(\varrho, v) = \mathfrak{f}\varrho v$ for some $\mathfrak{f} > 0$ (this is the so-called *damped barotropic compressible Euler equations*, also named *Euler equations with relaxation parameter* ε if $\mathfrak{f} = \varepsilon^{-1}$);
- or $A(\varrho, v) = \operatorname{div}_x(\mu(\varrho)(\nabla_x u + {}^t\nabla_x u)) + \nabla_x(\lambda(\varrho)\operatorname{div}_x u)$ for some smooth functions λ and μ satisfying $\mu > 0$ and $\lambda + 2\mu > 0$ (then, (2) is the *barotropic compressible Navier-Stokes equations*).

It is by now well understood that in the first situation (neither viscosity nor damping), smooth initial data generate a local-in-time solution that is likely to blow up after finite time (see e.g. [1, 34]) whereas in the second and third situations, small and sufficiently smooth perturbations of a constant density state

$$(3) \quad (\bar{\varrho}, 0) \quad \text{with } \bar{\varrho} > 0 \quad \text{and } P'(\bar{\varrho}) > 0$$

produce global strong solutions that are defined for all positive times.

The good diffusive properties of the barotropic compressible Navier-Stokes equations in the whole space \mathbb{R}^3 (and, more generally, of the full non-isothermal polytropic system) have been first observed by A. Matsumura and T. Nishida at the end of the 70ies. In [28], they established the global existence of strong solutions for $H^3(\mathbb{R}^3)$ perturbations of any constant state of type (3) (see [13] for a version of this result in the broader setting of ‘critical Besov spaces’). An important achievement in the study of general first order partially dissipative symmetric hyperbolic systems having both terms of order 0 and 2 has been made by S. Kawashima in 1984, in his PhD thesis [24]. There, he exhibited a rather simple sufficient condition that is nowadays called *the (SK) (meaning Shizuta-Kawashima) condition* for global existence of strong solutions in the neighborhood of linearly stable constant solutions. In the case where there is only a 0-order partially dissipative term, Condition (SK) exactly says that for the linearized system, *the intersection between the kernel of the 0-order term and the set of all eigenvectors of the symmetric first order term is reduced to $\{0\}$.*

A bit later, S. Shizuta and S. Kawashima in [33] observed that Condition (SK) is equivalent to the fact that, in the Fourier space, the real parts of all eigenvalues of the matrix of the linearized system about the reference solution are strictly negative and also to the existence of a *compensating function*. That compensating function comes into play for working out a functional that is equivalent to a Sobolev norm of high order and allows to recover the optimal dissipative properties of the system. In the same paper, the authors pointed out that, if in addition of being in a Sobolev space $H^s(\mathbb{R}^d)$ with large enough s , the discrepancy of the initial data to the reference constant solution \bar{V} belongs to some Lebesgue space $L^p(\mathbb{R}^d)$ with $p \in [1, 2]$, then the global solution V converges to \bar{V} in $L^2(\mathbb{R}^d)$ with the same decay rate as for the heat equation, namely $(1+t)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{2})}$, when t goes to infinity. Since then, more decay estimates have been proved under various assumptions in e.g. [5, 37, 40].

A number of more accurate results have been obtained since then for specific systems. For instance, T. Sideris *et al* [35] considered the three-dimensional compressible Euler equations with damping and Y. Zeng [43] studied a particular class of 4×4 nonlinear hyperbolic system with relaxation. General partially (0-order) dissipative systems have been investigated by S. Kawashima and W.-A. Yong in [25, 26] and by W.-A. Yong in [42], and adapted to second order partially diffusive operators by V. Giovangigli *et al* in [18, 19]. Recent works on general partially dissipative systems in the so-called critical functional framework (that will be recalled later in this text) have been performed by J. Xu and S. Kawashima [38, 39, 40].

It has also been observed by several authors that Condition (SK) is not necessary for the existence of global strong solutions. For instance, in [31], P. Qu and Y. Wang established a global existence result in the case where exactly one eigenvector violates Condition (SK). In this respect, one can also mention the paper by R. Bianchini and R. Natalini [6] that uses nonresonant bilinear forms, and the recent work [8] dedicated to the mathematical study of a model of mixture of compressible fluids.

The strength of Shizuta and Kawashima’s approach is that it does not require to compute explicitly the Green function of the linearized system under consideration. Although doing this calculation for the damped barotropic Euler equations presented above is not an issue, computing the Green kernel associated to the corresponding linearized system in the nonisothermal case is already more involved, and it soon becomes impossible for more cumbersome systems (like e.g. systems related to the description of plasma or radiative phenomena, see e.g [16]). As said before, having a ‘compensating function’ at hand allows to construct an energy functional that encodes the dissipative properties of the system.

In Shizuta and Kawashima's work however, this functional is not so explicit, that makes difficult, if not impossible, to track the dependency of the solution with respect to the parameters of the system, when applicable. Another limitation is that it only provides estimates on the whole solution, without supplying more accurate informations on the part of the solution which is expected to experience a better dissipation.

In [3], K. Beauchard and E. Zuazua took advantage of techniques that originate from Kalman control theory for linear ODEs so as to construct explicit Lyapunov functionals for general partially dissipative systems of order 1. They also pointed out the connection between Condition (SK) and the Kalman criterion for observability in the theory of linear ODEs (this was also noticed by D. Serre in his unpublished lecture notes [32]). To some extent, Beauchard and Zuazua's approach may be interpreted in the broader framework of hypo-ellipticity as presented by L. Hörmander in [22] or, much more recently, by C. Villani in [36]. To keep these notes as elementary and short as possible, we refrain from looking deeper into this direction, though.

Although it is not mentioned in the construction of a Lyapunov functional, Beauchard and Zuazua's approach provides for free compensating functions. Furthermore, the construction is elementary (it suffices to compute at most n powers of matrices) and easily localizable in the Fourier space. Hence, at the linear level, keeping track of the different behavior of the low and of the high frequencies of the solution is obvious. Their method further allows to handle some systems that do not satisfy Condition (SK) (but we shall not investigate this interesting point in these notes).

The present lecture notes aim at familiarizing the reader with the Beauchard-Zuazua approach and recent updates that originate from the thesis of T. Crin-Barat and were published in [10, 11, 12]. As our aim is not to provide the reader with an exhaustive theory of partially dissipative hyperbolic systems but rather to present a clear road map allowing him to tackle efficiently the study of systems of this type, we shall focus on the following 'academic' class of partially dissipative hyperbolic systems:

$$(4) \quad \partial_t V + \sum_{k=1}^d A^k(V) \partial_k V = \varepsilon^{-1} H(V).$$

Above, the (smooth) functions A^k ($k = 1, \dots, d$) and H are defined on some open subset \mathcal{O} of \mathbb{R}^n , and have range in the set of $n \times n$ real symmetric matrices, and in \mathbb{R}^n , respectively. The unknown $V = V(t, x)$ depends on the time variable $t \in \mathbb{R}_+$ and on the space variable $x \in \mathbb{R}^d$ ($d \geq 1$). We fix a constant solution $\bar{V} \in \mathcal{O}$ of (4) (hence $H(\bar{V}) = 0$). The system is supplemented with initial data $V_0 \in \mathcal{O}$ at time $t = 0$, that are sufficiently close to \bar{V} . Finally, the relaxation parameter ε is a given positive parameter that, except in Section 4, is taken equal to 1.

A basic example of a physical system in the above class is the compressible Euler equations with isentropic pressure law $P(\varrho) = a\varrho^\gamma$, if rewritten in terms of the (renormalized) sound speed

$$c \triangleq \frac{(\gamma A)^{\frac{1}{2}}}{\tilde{\gamma}}(\varrho)^{\tilde{\gamma}} \quad \text{with} \quad \tilde{\gamma} \triangleq \frac{\gamma - 1}{2}.$$

Indeed, the pair (c, v) then satisfies:

$$(5) \quad \begin{cases} \partial_t c + v \cdot \nabla c + \tilde{\gamma} c \operatorname{div} v = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t v + v \cdot \nabla v + \tilde{\gamma} c \nabla c + \varepsilon^{-1} v = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \end{cases}$$

Under the so-called Condition (SK) (presented in the next section) that is satisfied in particular by (5), we shall prove the existence of global strong solutions with ‘critical regularity’ for (4) in the neighborhood of any constant solution \bar{V} (see Theorems 2.1 and 2.2). Then, we shall obtain the strong convergence to \bar{V} in the long time asymptotics with explicit decay rates (Theorem 3.1). In Section 4, we shall investigate the strong relaxation limit, that is the convergence of the solutions of (4) to some limit system. Let us shortly explain what we mean in the simple case of the compressible Euler equations. Making the following ‘diffusive’ rescaling:

$$(\varrho, v)(t, x) = (\tilde{\varrho}, \varepsilon \tilde{v})(\varepsilon t, x),$$

we see that the pair $(\tilde{\varrho}, \tilde{v})$ satisfies:

$$\begin{cases} \partial_\tau \tilde{\varrho} + \operatorname{div}(\tilde{\varrho} \tilde{v}) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \varepsilon^2 \partial_\tau(\tilde{\varrho} \tilde{v}) + \varepsilon \operatorname{div}(\tilde{\varrho} \tilde{v} \otimes \tilde{v}) + \nabla(P(\tilde{\varrho})) + \tilde{\varrho} \tilde{v} = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \end{cases}$$

Hence, formally, if $\tilde{\varrho}$ and \tilde{v} tend to some functions N and w , then the second equation above yields

$$\nabla(P(N)) + Nw = 0$$

which, plugged in the mass conservation equation leads to the so-called porous media equation:

$$(6) \quad \partial_\tau N - \Delta(P(N)) = 0.$$

The rigorous justification of the convergence of the density to a solution of (6) has been first carried out by S. Junca and M. Rascle [23] in the one-dimensional case where specific techniques may be used. In the multi-dimensional case, the weak convergence and the strong convergence on bounded subsets of \mathbb{R}^d have been proved by J.-F. Coulombel and C. Lin in [27], and by Z. Wang and J. Xu in [41]. Results in the same spirit for a class of partially dissipative hyperbolic systems have been obtained by Y.-J. Peng and V. Wasiolek in [30]. The approach that is proposed in the present lecture notes allows to get the *strong convergence in the whole space with explicit convergence rates for suitable norms* when the relaxation parameter tends to zero not only for the Euler equations, but also for a class of partially hyperbolic systems (see Theorem 4.1).

It should be noted that, at the linear level, the method that has been originally proposed by K. Beauchard and E. Zuazua in [3] works exactly the same for partial differential operators of any order (and, more generally, for homogeneous Fourier multipliers) provided one of them is skew-symmetric and the other one, nonnegative. We will enrich this method by exhibiting a ‘damped mode’ for low frequencies, first introduced in [10] and [11] to the best of our knowledge. This the key to an optimal treatment of the low frequencies of the solution in a critical framework. With almost no additional effort, assuming a bit more integrability on the initial data (expressed in terms of negative Besov spaces like in the work [40] by J. Xu and S. Kawashima), and arguing essentially as in the paper by Y. Guo and Y. Wang [21], we will derive optimal time decay estimates, pointing out better decay for the high frequencies of the solution and for the damped mode. It turns out that adopting a critical approach with different levels of regularity for low and high frequencies also allows to keep track of the relaxation parameter ε just by suitable space/time rescaling. This substantially simplifies the study of the strong relaxation limit. Here again, having a damped mode at hand plays an essential role.

Except for our linear analysis, we here concentrate on first order hyperbolic symmetric systems with a partial dissipation term of order 0. The class that is considered contains the isentropic Euler equations with relaxation. We expect the whole strategy modified accordingly to be adaptable to hyperbolic-parabolic systems, to operators of any order and to more complex situations where the partially dissipative terms have mixed orders (see recent examples in [16] and [8]). It would also be of interest to study to what extent it may be adapted to situations where pseudo-differential operators depending on the space variable come into play. Since we used mostly Fourier analysis in our investigations, most of our results can be adapted to periodic boundary conditions in one or several directions, leading to the same statements in the first three sections (the strong relaxation limit studied in Section 4 may be different since the rescaling we used there changes the size of the periodic box). Handling ‘physical’ boundaries requires completely different tools, and we have no opinion on whether similar results are true or not.

The rest of these notes unfolds as follows. In the next section, we present Beauchard and Zuazua’s approach for linear partially dissipative hyperbolic systems with operators of any orders. This enables us to deduce quite easily global-in-time a priori estimates in ‘hybrid’ Besov spaces with different regularity exponents for low and high frequencies. We also exhibit a damped mode, the low frequencies of which satisfy better decay estimates and point out that, under additional structure conditions on the system, it is possible to use without much effort an L^p functional framework for the low frequencies. The following sections focus on the nonlinear system (4). In Section 2, we prove global-in-time results while time decay estimates are established in Section 3. In Section 4, we prove strong convergence results when the relaxation parameter ε tends to 0 for partially dissipative systems having the same structure as the isentropic compressible Euler equations with damping. A few technical results are recalled or proved in Appendix.

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1. THE LINEAR ANALYSIS

To better understand the difference between the three model situations corresponding to System (2), having first a look at the linearized equations about $(\bar{\varrho}, 0)$ is very informative. After suitable renormalization, the system to be considered reads:

$$(7) \quad \begin{cases} \partial_t a + \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t u + \nabla a + \kappa(-\Delta)^{\frac{\beta}{2}} u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \end{cases}$$

The above cases correspond to $(\kappa, \beta) = (0, 0)$, $(\kappa, \beta) = (\mathfrak{f}, 0)$ with $\mathfrak{f} > 0$ or $(\kappa, \beta) = (\mu(\bar{\varrho}), 2)$ (in the special situation $\lambda(\bar{\varrho}) + \mu(\bar{\varrho}) = 0$ the general case being similar), respectively.

If $\kappa = 0$ then System (7) is purely first order hyperbolic and no diffusion or dissipative phenomenon is expected whatsoever since all Sobolev norms are constant in time. In the multi-dimensional case, *dispersive* phenomena of wave equation type do exist, but they concern only the density and the potential part of the velocity (they will not be discussed here).

Let us focus on the case $\kappa > 0$ and $\beta \neq 1$ (not necessarily equal to 0 or 2). After suitable rescaling, one can then suppose that $\kappa = 1$. In the Fourier variable ξ corresponding to the

physical variable x , the above system (7) rewrites

$$(8) \quad \frac{d}{dt} \begin{pmatrix} \widehat{a} \\ \widehat{u} \end{pmatrix} + \begin{pmatrix} 0 & i\xi \\ i\xi & |\xi|^\beta \end{pmatrix} \begin{pmatrix} \widehat{a} \\ \widehat{u} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \xi \in \mathbb{R}^d.$$

In order to have some insight on the long-time behavior of the solution (a, u) , let us look at the eigenvalues of the $(d+1) \times (d+1)$ matrix of System (8). The eigenvalue $|\xi|^\beta$ appears with multiplicity $d-1$ (this corresponds to the ‘incompressible’ part of the velocity field). The remaining two eigenvalues $\lambda^\pm(\xi)$ capture the coupling between a and the ‘compressible’ part of u , and may be computed by considering the following 2×2 reduced system satisfied by a and $v \triangleq (-\Delta)^{-1/2} \operatorname{div} v$, namely, if $\kappa = 1$,

$$\begin{cases} \partial_t a + (-\Delta)^{1/2} v = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t v - (-\Delta)^{1/2} a + (-\Delta)^{\frac{\beta}{2}} v = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \end{cases}$$

The corresponding matrix in the Fourier space reads $\begin{pmatrix} 0 & |\xi| \\ -|\xi| & |\xi|^\beta \end{pmatrix}$.

Two different situations occur depending on whether β is smaller or greater than 1:

- The ‘dissipative’ situation $\beta < 1$:

$$\begin{aligned} \lambda^\pm(\xi) &= \frac{|\xi|^\beta}{2} \left(1 \pm \sqrt{1 - 4|\xi|^{2(1-\beta)}} \right) & \text{if } |\xi|^{1-\beta} < 1/2, \\ \lambda^\pm(\xi) &= \frac{|\xi|^\beta}{2} \left(1 \pm i\sqrt{4|\xi|^{2(1-\beta)} - 1} \right) & \text{if } |\xi|^{1-\beta} > 1/2. \end{aligned}$$

Observe that for $\xi \rightarrow 0$, we have $\lambda^+(\xi) \sim |\xi|^\beta$ (parabolic behavior similar to that of $(-\Delta)^{\beta/2}$) while $\lambda^-(\xi) \sim |\xi|^{2-\beta}$ (parabolic behavior of type $(-\Delta)^{1-\beta/2}$).

- The ‘diffusive’ situation $\beta > 1$:

$$\begin{aligned} \lambda^\pm(\xi) &= \frac{|\xi|^\beta}{2} \left(1 \pm i\sqrt{4|\xi|^{2(1-\beta)} - 1} \right) & \text{if } |\xi|^{\beta-1} < 1/2, \\ \lambda^\pm(\xi) &= \frac{|\xi|^\beta}{2} \left(1 \pm \sqrt{1 - 4|\xi|^{2(1-\beta)}} \right) & \text{if } |\xi|^{\beta-1} > 1/2. \end{aligned}$$

For $\xi \rightarrow \infty$, we have $\lambda^+(\xi) \sim |\xi|^\beta$ (parabolic behavior like for $(-\Delta)^{\beta/2}$) while $\lambda^-(\xi) \sim |\xi|^{2-\beta}$ (parabolic behavior similar to that of $(-\Delta)^{1-\beta/2}$).

At the linear level, the damped Euler equations and the compressible Navier-Stokes equations correspond to the dissipative and diffusive situations, respectively. We observe that, in the two cases, the whole solution decays to 0 with a decay rate that depends on $|\xi|$ although there is no damping term in the linearized mass equation. Note however that, depending on whether $\beta < 1$ or $\beta > 1$, the behavior of the low and high frequencies of the solution is exchanged.

Let us revert to our model system (4) with $\varepsilon = 1$ for simplicity, namely

$$(9) \quad \partial_t V + \sum_{k=1}^d A^k(V) \partial_k V = H(V).$$

Let us fix a constant solution \bar{V} of (9) (that is, $\bar{V} \in \mathcal{O}$ satisfies $H(\bar{V}) = 0$) and make the following structure assumptions on the system:

- (H1) For all $V \in \mathcal{O}$, the matrices $A^k(V)$ are real symmetric;
- (H2) The spectrum of $DH(\bar{V})$ is included in the set $\{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$.

In the case $H \equiv 0$ (no dissipation at all) smooth solutions, even small ones, may blow up after finite time. At the exact opposite, if the spectrum of $DH(\bar{V})$ is included in the set $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ then it is not difficult to show that small perturbations of \bar{V} in the Sobolev space H^s with $s > 1 + d/2$ generate global strong solutions that tend exponentially fast to \bar{V} when time goes to infinity. We here address the intermediate situation where some eigenvalues of $DH(\bar{V})$ vanish. For expository purpose, we assume that H is linear and has the block structure:

$$(10) \quad H(V) = \begin{pmatrix} 0 \\ -L_2(V_2 - \bar{V}) \end{pmatrix} \quad \text{with} \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

where $V_1 \in \mathbb{R}^{n_1}$, $V_2 \in \mathbb{R}^{n_2}$ (with $n_1 + n_2 = n$) and $L_2 : \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_2}$ is linear invertible and such that $L_2 + {}^tL_2$ is definite positive. Additional structure assumptions on L_2 and on the matrices A^k will be specified later on.

1.1. Reduction of the problem. Denoting $Z \triangleq V - \bar{V}$ and $LZ \triangleq -H(\bar{V} + Z)$ (with H as in (10)), the system for Z reads

$$(11) \quad \partial_t Z + \sum_{k=1}^d A^k(\bar{V} + Z) \partial_k Z + LZ = 0,$$

and the corresponding linearized system is thus

$$(12) \quad \partial_t Z + \sum_{k=1}^d \bar{A}^k \partial_k Z + LZ = F \quad \text{with} \quad \bar{A}^k \triangleq A^k(\bar{V}) \quad \text{for} \quad k = 1, \dots, d.$$

In the Fourier space, the above system recasts in

$$\partial_t \widehat{Z} + i \sum_{k=1}^d \bar{A}^k \xi_k \widehat{Z} + L \widehat{Z} = \widehat{F}.$$

The symmetry of the matrices \bar{A}^j ensures that for all $\xi \in \mathbb{R}^d$, the matrix

$$(13) \quad A(\xi) \triangleq i \sum_{k=1}^d \bar{A}^k \xi_k$$

is skew Hermitian, while the symmetric part of L is nonnegative. Denoting by $A(D)$ (resp. $B(D)$) the Fourier multiplier of symbol¹ A (resp. L), System (12) rewrites

$$(14) \quad \partial_t Z + A(D)Z + B(D)Z = F.$$

The analysis we present below is valid in the more general situation where:

- $A(D)$ is a homogeneous (matrix-valued) Fourier multiplier of degree α that satisfies

$$(15) \quad \operatorname{Re}(A(\omega)\eta \cdot \eta) = 0 \quad \text{for all} \quad \omega \in \mathbb{S}^{d-1} \quad \text{and} \quad \eta \in \mathbb{C}^n,$$

where \cdot designates the Hermitian scalar product in \mathbb{C}^n ,

- $B(D)$ is an homogeneous (matrix-valued) Fourier multiplier of degree β , such that, for some positive real number κ ,

$$(16) \quad \operatorname{Re}(B(\omega)\eta \cdot \eta) \geq \kappa |B(\omega)\eta|^2 \quad \text{for all} \quad \omega \in \mathbb{S}^{d-1} \quad \text{and} \quad \eta \in \mathbb{C}^n.$$

¹Throughout the text, we agree that $A(D) \triangleq \mathcal{F}^{-1}A\mathcal{F}$, where \mathcal{F} stands for the Fourier transform with respect to the variable x .

As a first example, if one considers the linearized damped compressible Euler equations about $(\varrho, v) = (1, 0)$ in the case $P'(1) = 1$, namely

$$(17) \quad \begin{cases} \partial_t a + \operatorname{div} u = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t u + \nabla a + \mathfrak{f}u = g & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \end{cases}$$

then we have $n_1 = 1$, $n_2 = d$, and the Fourier multipliers A and B read:

$$A(\xi) = i \begin{pmatrix} 0 & \xi \\ t\xi & 0 \end{pmatrix} \quad \text{and} \quad B(\xi) = \mathfrak{f} \begin{pmatrix} 0 & 0 \\ 0 & I_d \end{pmatrix}.$$

They are of order 1 and 0, respectively. Clearly, (15) holds true, as well as (16) with $\kappa = \mathfrak{f}^{-1}$.

As a second example, consider the linearized compressible Navier-Stokes equations about $(\varrho, v) = (1, 0)$. Denoting $\bar{\mu} \triangleq \mu(\bar{\varrho})$ and $\bar{\lambda} \triangleq \lambda(\bar{\varrho})$, they read

$$(18) \quad \begin{cases} \partial_t a + \operatorname{div} u = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t u + \nabla a - \mu \Delta u - (\lambda + \mu) \operatorname{div} u = g & \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \end{cases}$$

We still have $n_1 = 1$, $n_2 = d$, but the Fourier multipliers A and B now read:

$$A(\xi) = i \begin{pmatrix} 0 & \xi \\ t\xi & 0 \end{pmatrix} \quad \text{and} \quad B(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & \bar{\mu}|\xi|^2 I_d + (\bar{\lambda} + \bar{\mu})\xi \otimes \xi \end{pmatrix}.$$

They are of order 1 and 2, respectively. Both properties (15) and (16) hold true (with κ depending on $\bar{\lambda}$ and $\bar{\mu}$) provided $\bar{\mu} > 0$ and $\bar{\lambda} + 2\bar{\mu} > 0$.

System (14) may be solved by means of Duhamel's formula:

$$(19) \quad Z(t) = T(t)Z(0) + \int_0^t T(t-\tau)F(\tau) d\tau,$$

where $(T(t))_{t \geq 0}$ stands for the semi-group associated to operator $-(A+B)(D)$.

The value of $T(t)$ may be computed by going into the Fourier space. Indeed, denote by \widehat{Z} the Fourier transform of Z with respect to x , and by ξ the corresponding Fourier variable. Then, in the case $F = 0$, System (14) rewrites:

$$\partial_t \widehat{Z} + E(\xi)\widehat{Z} = 0 \quad \text{with} \quad E(\xi) \triangleq A(\xi) + B(\xi).$$

Hence $\widehat{Z}(t, \xi) = \exp(-E(\xi)t)\widehat{Z}_0(\xi)$. In other words, we have $T(t) = \exp(-E(D)t)$.

In what follows we set for all $\omega \in \mathbb{S}^{d-1}$ and $\rho > 0$,

$$(20) \quad A(\xi) = \rho^\alpha A_\omega \quad \text{and} \quad B(\xi) = \kappa^{-1} \rho^\beta B_\omega \quad \text{with} \quad \xi = \rho\omega.$$

With this notation, we have

$$(21) \quad \widehat{Z}(t, \xi) = \widehat{Z}(0, \xi) \exp\left(-\frac{t\rho^\beta}{\kappa}(\kappa\rho^{\alpha-\beta}A_\omega + B_\omega)\right).$$

Making the change of variable $\tau \triangleq (t\rho^\beta)/\kappa$ and $r \triangleq \kappa\rho^{\alpha-\beta}$, we discover that $z(\tau) \triangleq \widehat{Z}(t, \xi)$ is the solution to

$$(22) \quad z' + E_{r,\omega}z = 0 \quad \text{with} \quad E_{r,\omega} \triangleq rA_\omega + B_\omega$$

since we have

$$z(\tau) = z(0) \exp(-\tau E_{r,\omega}).$$

Hence, the case $\alpha = 1$, $\beta = 0$ and $\kappa = 1$, is generic at the linear level.

1.2. Derivation of a Lyapunov functional. The long time behavior of z is closely connected to the signs of the real part of the eigenvalues of the matrix $E_{r,\omega}$ defined in (22). The method proposed by K. Beauchard and E. Zuazua in [3] (see also [14, 15]), that is inspired by Kalman's control theory for linear ODEs supplies a simple way for constructing an explicit Lyapunov functional and a dissipation term altogether *without computing the eigenvalues*.

To explain the construction, fix some $r > 0$ and $\omega \in \mathbb{S}^{d-1}$, and consider the ODE (22) satisfied by z . Combining the assumptions (15) and (16) with the renormalization (20) ensures that

$$(23) \quad \operatorname{Re}((A_\omega \eta) \cdot \eta) = 0 \quad \text{and} \quad \operatorname{Re}((B_\omega \eta) \cdot \eta) \geq |B_\omega \eta|^2 \quad \text{for all } (\omega, \eta) \in \mathbb{S}^{d-1} \times \mathbb{C}^n.$$

Hence, taking the Hermitian product in \mathbb{C}^n of (22) with z and keeping the real part yields

$$(24) \quad \frac{1}{2} \frac{d}{dt} |z|^2 + |B_\omega z|^2 \leq 0.$$

If B_ω has rank strictly smaller than n , then the above inequality does not ensure decay of all the components of z (even though this decay exists whenever $r > 0$ and $\omega \in \mathbb{S}^{d-1}$ are such that the real parts of all the eigenvalues of the matrix $E_{r,\omega}$ are positive). To recover the decay (if any) for the 'missing components' of the solution, one can start with the identity

$$(B_\omega z)' + (rB_\omega A_\omega + B_\omega^2)z = 0.$$

Hence, taking the Hermitian product with BAz (we drop the index ω for better readability), we obtain

$$Bz' \cdot BAz + r|BAz|^2 + B^2z \cdot BAz = 0.$$

Similarly, we have

$$Bz \cdot BAz' + rBz \cdot BA^2z + Bz \cdot B^2Az = 0,$$

whence

$$\frac{d}{dt} (Bz \cdot BAz) + r|BAz|^2 + B^2z \cdot BAz + Bz \cdot B^2Az = -rBz \cdot BA^2z.$$

Remembering (24) and using several times the obvious inequality

$$2 \operatorname{Re}(a \cdot b) \leq K|a|^2 + K^{-1}|b|^2$$

with suitable values of K , we discover that one can find some ε_1 (that can be taken arbitrarily small) such that

$$(25) \quad \frac{d}{dt} \left(|z|^2 + \varepsilon_1 \min(r, r^{-1}) \operatorname{Re}(Bz \cdot BAz) \right) + |Bz|^2 + \varepsilon_1 \min(1, r^2) |BAz|^2 \\ \leq C\varepsilon_1 \min(1, r^2) |BA^2z|^2.$$

In the case $BA^2 \neq 0$, we need (at least) one more relation to handle the term in the right-hand side. For that, one can start from the equation

$$(BAz)' + (rBA^2 + BAB)z = 0$$

and take the Hermitian scalar product with BA^2z , adding up the resulting identity multiplied by a small enough ε_2 to (25), then iterate the procedure. The fundamental observation

of Beauchard and Zuazua in [3] is that Cayley-Hamilton theorem ensures the existence of complex numbers c_0, \dots, c_{n-1} so that

$$A^n = \sum_{k=0}^{n-1} c_k A^k.$$

Consequently, one can end the process after at most n steps. In the end, we get positive parameters $\varepsilon_0 = 1$ and $\varepsilon_1, \dots, \varepsilon_{n-1}$ (that are defined inductively and can be taken arbitrarily small) such that for all $\omega \in \mathbb{S}^{d-1}$ and $r > 0$, we have

$$(26) \quad \frac{d}{dt} L_{r,\omega}(z) + \frac{\min(1, r^2)}{2} \sum_{\ell=0}^{n-1} \varepsilon_\ell |B_\omega A_\omega^\ell z|^2 \leq 0$$

$$\text{with } L_{r,\omega}(z) \triangleq |z|^2 + \min(r, r^{-1}) \sum_{\ell=1}^{n-1} \varepsilon_\ell \operatorname{Re}(B_\omega A_\omega^{\ell-1} z \cdot B_\omega A_\omega^\ell z)$$

and, additionally,

$$(27) \quad \frac{1}{2}|z|^2 \leq L_{r,\omega}(z) \leq 2|z|^2.$$

Consequently, denoting $N_\omega \triangleq \inf \left\{ \sum_{\ell=0}^{n-1} \varepsilon_\ell |B_\omega A_\omega^\ell x|^2, x \in \mathbb{S}^{d-1} \right\}$, we conclude from (26) and (27) that

$$(28) \quad L_{r,\omega}(\tau) \leq e^{-\frac{1}{4} \min(1, r^2) N_\omega \tau} L_{r,\omega}(0) \quad \omega \in \mathbb{S}^{d-1}, \quad r > 0.$$

In the particular case where

$$(29) \quad N_\omega > 0 \quad \text{for all } \omega \in \mathbb{S}^{d-1},$$

(the only situation that will be considered in these notes) then N_ω is actually bounded away from zero owing to the compactness of the sphere. Hence, (28) implies that there exists a positive constant c such that for all $r > 0$ and $\omega \in \mathbb{S}^{d-1}$, we have

$$L_{r,\omega}(\tau) \leq e^{-2c \min(1, r^2) \tau} L_{r,\omega}(0), \quad \tau \geq 0.$$

Then, using once more (27) and reverting to the original unknown \widehat{Z} , we conclude that

$$(30) \quad |\widehat{Z}(t, \xi)| \leq 2e^{-c \min(\kappa^{-1} |\xi|^\beta, \kappa |\xi|^{2\alpha-\beta}) t} |\widehat{Z}_0(\xi)|.$$

In other words, if (29) holds then:

- either $\alpha > \beta$, and we are in a partially dissipative regime similar to that of linearized compressible Euler equations,
- or $\alpha < \beta$, and we are in a partially diffusive regime analogous to that of the linearized compressible Navier-Stokes equations.

It has been pointed out in [3] that (29) is equivalent to the Shizuta-Kawashima condition. The following lemma stresses the link between those two conditions, the strict dissipativity of System (11) and Kalman's condition for observability.

Lemma 1.1. *Let A and B be two $n \times n$ complex valued matrices. Assume that A is skew-symmetric in the meaning of (15) and that B is nonnegative in the sense of (16). The following properties are equivalent:*

- (1) For all positive $\varepsilon_0, \dots, \varepsilon_{n-1}$, we have $\sum_{\ell=0}^{n-1} \varepsilon_\ell |BA^\ell \eta|^2 > 0$ for all $\eta \in \mathbb{S}^{n-1}$.
- (2) We have the Kalman rank property, namely the $n^2 \times n$ matrix $\begin{pmatrix} B \\ BA \\ \dots \\ BA^{n-1} \end{pmatrix}$ has rank equal to n .
- (3) The (SK) condition holds true, namely the intersection between $\ker B$ and the linear space of all eigenvectors of A is reduced to $\{0\}$.
- (4) All eigenvalues of $A + B$ have positive real parts.

Proof. The equivalence between the first three items is basic linear algebra (see details in e.g. [3]), while Inequality (28) (with $A_\omega = A$, $B_\omega = B$ and $r = 1$) ensures equivalence with the last item. \square

As an example, let us again consider the linearized compressible Euler equations (17). As said before, (15) and (16) are satisfied with $\alpha = 1$, $\beta = 0$, $\kappa = \mathfrak{f}^{-1}$. Furthermore, we have

$$A_\omega = i \begin{pmatrix} 0 & \omega \\ t_\omega & 0 \end{pmatrix} \quad \text{and} \quad B_\omega = \begin{pmatrix} 0 & 0 \\ 0 & I_d \end{pmatrix}, \quad \text{so that} \quad B_\omega A_\omega = i \begin{pmatrix} 0 & 0 \\ t_\omega & 0 \end{pmatrix}.$$

Hence $\begin{pmatrix} B_\omega \\ B_\omega A_\omega \end{pmatrix}$ has rank $d + 1$ and Kalman rank condition is thus satisfied, which gives eventually

$$|\widehat{Z}(t, \xi)| \leq 2e^{-c \min(\mathfrak{f}, \mathfrak{f}^{-1}) |\xi|^2 t} |\widehat{Z}_0(\xi)| \quad \text{for all } \xi \in \mathbb{R}^d \quad \text{and} \quad t \geq 0.$$

Since we do not need higher powers of A_ω to ensure the Kalman rank condition, one can suspect that one can restrict the sum in the definition of the Lyapunov function $L_{r, \omega}$ to only one term ($\ell = 1$). Now, the reader may observe by direct computation that $B_\omega A_\omega^2 = A_\omega^2 B_\omega$. Hence

$$-B_\omega z \cdot B_\omega A_\omega^2 z = -B_\omega z \cdot A_\omega^2 B_\omega z \leq C |B_\omega z|^2$$

and taking ε_1 sufficiently small in (25) allows to just have

$$\frac{d}{dt} \left(|z|^2 + \varepsilon_1 \min(r, r^{-1}) \operatorname{Re} (B_\omega z \cdot B_\omega A_\omega z) \right) + |B_\omega z|^2 + \varepsilon_1 \min(1, r^2) |B_\omega A_\omega z|^2 \leq 0.$$

One can be more explicit : since $z = (\widehat{a}, \widehat{u})$ and $\xi = r\omega$, we have

$$\min(r, r^{-1}) B_\omega z \cdot B_\omega A_\omega z = \min(1, |\xi|^{-2}) (\widehat{u} \cdot (i\xi) \widehat{a}) = \min(1, |\xi|^{-2}) (\widehat{u} \cdot (\widehat{D}a)).$$

Hence, we conclude that the Lyapunov functional is of the form

$$\mathcal{L}(\xi) = |\widehat{a}|^2 + |\widehat{u}|^2 + \varepsilon_1 \min(1, |\xi|^{-2}) \operatorname{Re} (\widehat{u} \cdot (\widehat{D}a)).$$

Combining with Fourier-Plancherel theorem, one can conclude that in order to recover the full dissipative properties of the linearized compressible Euler equations, it suffices to consider the functional

$$\|a\|_{L^2}^2 + \|u\|_{L^2}^2 + \varepsilon_1 \int_{\mathbb{R}^d} u \cdot (\operatorname{Id} - \Delta)^{-1} \nabla a \, dx$$

with suitably small ε_1 or, rather, spectrally localized versions of it.

Similar computations are valid for the linearized compressible Navier-Stokes equations (18). The reader may find more details in [13].

1.3. Derivation of a priori estimates. Let us assume from now on that $\kappa = 1$, $\alpha = 1$ and $\beta = 1$ in (14) (since the general case $\alpha \neq \beta$ reduces to that one). Recall Duhamel's formula (19). Combining with (30), we get

$$|\widehat{Z}(t, \xi)| \leq 2 \left(e^{-c \min(1, |\xi|^2)t} |\widehat{Z}_0(\xi)| + \int_0^t e^{-c \min(1, |\xi|^2)(t-\tau)} |\widehat{F}(\tau, \xi)| d\tau \right).$$

Clearly, if one wants to get optimal estimates then low and high frequencies have to be treated differently. To proceed, we shall actually use a more accurate decomposition of the Fourier space, namely a dyadic homogeneous Littlewood-Paley decomposition $(\dot{\Delta}_j)_{j \in \mathbb{Z}}$ defined by $\dot{\Delta}_j \triangleq \varphi(2^{-j}D)$. Here, φ is a smooth nonnegative function on \mathbb{R}^d , supported in (say) the annulus $\{\xi \in \mathbb{R}^d, 3/4 \leq |\xi| \leq 8/3\}$ and satisfying

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \neq 0.$$

By construction, $\dot{\Delta}_j$ is a localization operator in the vicinity of frequencies of magnitude 2^j . Since $\dot{\Delta}_j$ commutes with any Fourier multiplier, each $Z_j \triangleq \dot{\Delta}_j Z$ satisfies (14) with source term $F_j \triangleq \dot{\Delta}_j F$ and initial data $Z_{0,j} \triangleq \dot{\Delta}_j Z_0$. Therefore, we have

$$Z_j(t) = T(t)Z_{0,j} + \int_0^t T(t-\tau)F_j(\tau) d\tau,$$

whence, as $|\xi| \simeq 2^j$ on $\text{Supp } \widehat{Z}_j$, we have (changing slightly c if needed),

$$|\widehat{Z}_j(t, \xi)| \leq 2 \left(e^{-c \min(1, 2^{2j})t} |\widehat{Z}_{j,0}(\xi)| + \int_0^t e^{-c \min(1, 2^{2j})(t-\tau)} |\widehat{F}_j(\tau, \xi)| d\tau \right).$$

Consequently, after taking the $L^2(\mathbb{R}^d)$ norm of both sides, then using Minkowski inequality and Fourier-Plancherel theorem, we end up with:

$$\|Z_j(t)\|_{L^2} \leq 2 \left(e^{-c \min(1, 2^{2j})t} \|Z_{0,j}\|_{L^2} + \int_0^t e^{-c \min(1, 2^{2j})(t-\tau)} \|F_j(\tau)\|_{L^2} d\tau \right),$$

whence

$$(31) \quad \|Z_j(t)\|_{L^2} + c \min(1, 2^{2j}) \int_0^t \|Z_j\|_{L^2} d\tau \leq 2 \left(\|Z_{0,j}\|_{L^2} + \int_0^t \|F_j(\tau)\|_{L^2} d\tau \right).$$

At this stage, two important observations are in order. First, note that

$$\|Z\|_{\dot{H}^s} \simeq \left(\sum_{j \in \mathbb{Z}} 2^{2js} \|Z_j\|_{L^2}^2 \right)^{1/2}.$$

Hence, in order to get a Sobolev estimate of Z , it suffices to multiply (31) by 2^{js} then to perform an ℓ^2 -summation on $j \in \mathbb{Z}$. However, the second term of (31) will not exactly give an estimate in some space $L^1(0, t; \dot{H}^\sigma)$ since the time integration has been performed *before* the summation with respect to j : one ends up in one of the Chemin-Lerner (or 'tilde') spaces that have been introduced in [9]. They turn out to be delicate to manipulate and not adapted to the critical regularity setting we have in mind.

The second observation is that, owing to the factor $\min(1, 2^{2j})$, in order to track as much information as possible, it is suitable to work with *different* regularity exponents for low and high frequencies.

Putting the two observations together, this motivates us to multiply (31) by 2^{js} with a different value of the 'regularity exponent' s for negative and positive j 's, then to perform

an ℓ^1 summation with respect to j . The advantage of ℓ^1 summation – that corresponds to Besov norms *with last index 1* – is that one can freely exchange time integration and summation on j . Taking into account the possible difference of regularity between the low and high frequencies leads us to introduce for all pair $(s, s') \in \mathbb{R}^2$ the *hybrid Besov space* $\tilde{B}_{2,1}^{s,s'}$, that is the set of all tempered distributions z such that

$$(32) \quad \|z\|_{\tilde{B}_{2,1}^{s,s'}} \triangleq \sum_{j<0} 2^{js} \|\dot{\Delta}_j z\|_{L^2} + \sum_{j \geq 0} 2^{js'} \|\dot{\Delta}_j z\|_{L^2} < \infty \quad \text{and} \quad \lim_{j \rightarrow -\infty} \|\chi(2^{-j}D)z\|_{L^\infty} = 0.$$

Above, χ stands for a compactly supported smooth function on \mathbb{R}^d such that $\chi(0) = 1$, and the condition on $\chi(2^{-j}D)z$ implies that z has to tend to 0 at ∞ in the sense of tempered distributions². Classical homogenous Besov spaces correspond to $s = s'$ and will be denoted by $\dot{B}_{2,1}^s$.

In what follows, it will be sometimes convenient to use the following notation for all $\sigma \in \mathbb{R}$:

$$\|z\|_{\dot{B}_{2,1}^\sigma}^\ell \triangleq \sum_{j<0} 2^{j\sigma} \|\dot{\Delta}_j z\|_{L^2} \quad \text{and} \quad \|z\|_{\dot{B}_{2,1}^\sigma}^h \triangleq \sum_{j \geq 0} 2^{j\sigma} \|\dot{\Delta}_j z\|_{L^2}.$$

Even though most of the functions we shall consider here will have range in the set of vectors or even matrices, we shall keep the same notation for Besov spaces pertaining to this case.

Now, multiplying (31) by 2^{js} (resp. $2^{js'}$) for $j \leq 1$ (resp. $j \geq 0$) and summing up on $j \leq 1$ (resp. $j \geq 0$) leads to

$$(33) \quad \|Z(t)\|_{\dot{B}_{2,1}^s}^\ell + \int_0^t \|Z\|_{\dot{B}_{2,1}^{s+2}}^\ell d\tau \leq 2 \left(\|Z_0\|_{\dot{B}_{2,1}^s}^\ell + \int_0^t \|F\|_{\dot{B}_{2,1}^s}^\ell d\tau \right),$$

$$(34) \quad \|Z(t)\|_{\dot{B}_{2,1}^{s'}}^h + \int_0^t \|Z\|_{\dot{B}_{2,1}^{s'}}^h d\tau \lesssim 2 \left(\|Z_0\|_{\dot{B}_{2,1}^{s'}}^h + \int_0^t \|F\|_{\dot{B}_{2,1}^{s'}}^h d\tau \right).$$

Hence, putting together those two inequalities yields

$$(35) \quad \|Z(t)\|_{\tilde{B}_{2,1}^{s,s'}} + \int_0^t (\|Z\|_{\dot{B}_{2,1}^{s+2}}^\ell + \|Z\|_{\dot{B}_{2,1}^{s'}}^h) d\tau \leq 2 \left(\|Z_0\|_{\tilde{B}_{2,1}^{s,s'}} + \int_0^t \|F\|_{\tilde{B}_{2,1}^{s,s'}} d\tau \right).$$

Since a part of the solution experiences direct dissipation, one can suspect the low frequency integrability we get in this way to be not optimal. Recovering better integrability for a part of the solution is the goal of the next subsection.

1.4. The damped mode. Assume that the system has an orthogonal block structure, that is independent of the frequency, namely

$$\text{rank} B_\omega \oplus \text{Ker} B_\omega = \mathbb{C}^n \quad \text{for all } \omega \in \mathbb{S}^{n-1},$$

with $M \triangleq \text{Ker} B_\omega$ independent of ω .

Denote by \mathcal{P} the orthogonal projector onto M^\perp and set

$$(36) \quad W \triangleq \mathcal{P}(A + B)(D)Z.$$

Since \mathcal{P} and B commute, we get the following equation for W :

$$\partial_t W + B(D)W = \mathcal{P}(A + B)(D)F - \mathcal{P}A(D)(A + B)(D)Z.$$

²This is a way to rule out polynomials from homogeneous Besov spaces, otherwise one would have to work modulo polynomials which is not suitable when studying PDEs.

Because

$$\mathcal{P}A(D)B(D)Z = \mathcal{P}A(D)\mathcal{P}B(D)Z = \mathcal{P}A(D)W - (\mathcal{P}A(D))^2Z,$$

this may be rewritten:

$$(37) \quad \partial_t W + B(D)W = \mathcal{P}(A+B)(D)F - \mathcal{P}A(D)W + (\mathcal{P}A(D))^2Z - \mathcal{P}A^2(D)Z.$$

As $A(D)$ and $B(D)$ are of order 1 and 0, respectively, multipliers of orders 1 and 2, act on W and Z in the right-hand side. Hence the low frequencies of the corresponding terms are expected to be negligible compared to the left-hand side of (37).

To make this heuristics rigorous, let us look at the equation for $W_j \triangleq \hat{\Delta}_j W$, namely

$$(38) \quad \partial_t W_j + B(D)W_j = \mathcal{P}(A+B)(D)F_j - \mathcal{P}A(D)W_j + (\mathcal{P}A(D))^2Z_j - \mathcal{P}A^2(D)Z_j.$$

Taking the Hermitian scalar product in \mathbb{C}^n with W_j , using (16), the fact that $B(D)$ is 0-order and that $A(D)$ is 1-st order yields

$$\frac{1}{2} \frac{d}{dt} |\widehat{W}_j|^2 + |\widehat{W}_j|^2 \leq C \left((1 + |\xi|) |\widehat{F}_j| + |\xi| |\widehat{W}_j| + |\xi|^2 |\widehat{Z}_j| \right) |\widehat{W}_j|.$$

Hence, integrating on \mathbb{R}^d and taking advantage of the Fourier-Plancherel theorem yields:

$$\frac{1}{2} \frac{d}{dt} \|W_j\|_{L^2}^2 + \|W_j\|_{L^2}^2 \leq C \|W_j\|_{L^2} \left((1 + 2^j) \|F_j\|_{L^2} + 2^j \|W_j\|_{L^2} + 2^{2j} \|Z_j\|_{L^2} \right)$$

from which we eventually get for all $t \geq 0$ and $j \in \mathbb{Z}$, owing to Lemma A.1,

$$\begin{aligned} \|W_j(t)\|_{L^2} + \int_0^t \|W_j\|_{L^2} d\tau &\leq \|W_j(0)\|_{L^2} \\ &\quad + C(1 + 2^j) \int_0^t \|F_j\|_{L^2} d\tau + C2^j \int_0^t \|W_j\|_{L^2} d\tau + C2^{2j} \int_0^t \|Z_j\|_{L^2} d\tau. \end{aligned}$$

Therefore, if we multiply by 2^{js} and sum up on $j \leq j_0$ with j_0 chosen so that $C2^{j_0} \leq 1/2$, then we end up with

$$\begin{aligned} \sum_{j \leq j_0} 2^{js} \|W_j(t)\|_{L^2} + \frac{1}{2} \int_0^t \sum_{j \leq j_0} 2^{js} \|W_j\|_{L^2} d\tau &\leq \sum_{j \leq j_0} 2^{js} \|W_j(0)\|_{L^2} \\ &\quad + C \int_0^t \sum_{j \leq j_0} 2^{js} \|F_j\|_{L^2} d\tau + C \sum_{j \leq j_0} \int_0^t 2^{j(s+2)} \|Z_j\|_{L^2} d\tau. \end{aligned}$$

The last term may be controlled by the data according to (33). Furthermore, $\|W_j\|_{L^2} \lesssim \|Z_j\|_{L^2}$ for all $j < 0$, and $2^{j(s+2)} \simeq 2^{js}$ for $j_0 \leq j < 0$. Hence the above inequality still holds if one sums up to $j = 0$. In the end, this allows us to get the following additional bound:

$$\|W(t)\|_{\dot{B}_{2,1}^s}^\ell + \int_0^t \|W\|_{\dot{B}_{2,1}^s}^\ell d\tau \lesssim \|Z_0\|_{\dot{B}_{2,1}^s}^\ell + \int_0^t \|F\|_{\dot{B}_{2,1}^s}^\ell d\tau.$$

Let us finally look at the part of Z that undergoes direct dissipation, namely $Z_2 \triangleq \mathcal{P}Z$. We claim that, as expected, the low frequencies of Z_2 have better time integrability than the overall solution Z . Indeed, observing that $B(D)Z_2 = W - \mathcal{P}A(D)Z$ and that $\mathcal{P}B(D)$ (restricted to functions defined on M) is invertible, we may write

$$Z_2 = ((\mathcal{P}B)(D))^{-1}W - ((\mathcal{P}B)(D))^{-1}\mathcal{P}A(D)Z.$$

Hence, since $(\mathcal{P}B)(D)$ (resp. $A(D)$) is a 0-order (resp. 1-st order) Fourier multiplier, we may write

$$\|Z_2\|_{\dot{B}_{2,1}^{s+1}}^\ell \lesssim \|W\|_{\dot{B}_{2,1}^{s+1}}^\ell + \|Z\|_{\dot{B}_{2,1}^{s+2}}^\ell \quad \text{and} \quad \|Z_2\|_{\dot{B}_{2,1}^s}^\ell \lesssim \|W\|_{\dot{B}_{2,1}^s}^\ell + \|Z\|_{\dot{B}_{2,1}^{s+1}}^\ell.$$

Then, remembering (33) and using Hölder inequality and interpolation in Besov spaces when needed yields

$$\|Z_2\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^s)}^\ell + \|Z_2\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{s+1})}^\ell \lesssim \|Z_0\|_{\dot{B}_{2,1}^s}^\ell + \int_0^t \|F\|_{\dot{B}_{2,1}^s}^\ell d\tau.$$

This has to be compared by the following (optimal) inequality for Z :

$$\|Z\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{s+1})}^\ell + \|Z_2\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{s+2})}^\ell \lesssim \|Z_0\|_{\dot{B}_{2,1}^s}^\ell + \int_0^t \|F\|_{\dot{B}_{2,1}^s}^\ell d\tau.$$

1.5. An L^p approach. In this part, we are going to show that under slightly stronger structure assumptions³ on the linear system (12) than those that have been made so far, it is possible to bound the low frequencies of the solution on functional spaces built on L^p for any $p \in [1, \infty]$. This unusual setting is in sharp contrast with the non dissipative case. In fact, as pointed out by P. Brenner in [7], apart from the notable exception of the transport equation, ‘most’ first order ‘purely’ hyperbolic systems are ill-posed in L^p if $p \neq 2$. It turns out that for nonlinear partially dissipative systems satisfying the structure assumptions of this part, it is also possible to use, at least partially, an L^p type framework (see details in [10, 12]). This offers one more degree of freedom in the choice of solutions spaces allowing not only to prescribe weaker smallness conditions for global well-posedness, but also to get more accurate informations on the qualitative properties of the constructed solutions.

In order to proceed, let us assume without loss of generality that $M = \mathbb{R}^{n_1} \times \{0\}$ and decompose $Z \in \mathbb{R}^n$ into $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$. For expository purpose, further assume that there is no source term ($F = 0$). Then, System (11) may be rewritten by blocks as follows:

$$(39) \quad \frac{d}{dt} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \begin{pmatrix} A_{11}(D) & A_{12}(D) \\ A_{21}(D) & A_{22}(D) \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ B_{22}(D)Z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the 0-order Fourier multiplier $B_{22}(D)$ has symbol in $\mathcal{M}_{n_2}(\mathbb{R})$, and so on.

In the spirit of the computations of the previous paragraph, let us introduce

$$(40) \quad W \triangleq Z_2 + (B_{22}^{-1}A_{21})(D)Z_1 + (B_{22}^{-1}A_{22})(D)Z_2.$$

This definition of a damped mode is consistent with the one we had before: we just applied to (36) the 0-order operator $(B_{22}(D))^{-1}$ that corresponds to the inverse of $\mathcal{P}B(D)$ restricted to M . Now, we note that

$$\partial_t Z_2 + B_{22}(D)W = 0$$

and that from the definition of Z , we have

$$\partial_t W + B_{22}(D)W = (B_{22}^{-1}A_{11})(D)\partial_t Z_1 + (B_{22}^{-1}A_{22})(D)\partial_t Z_2.$$

Hence, using System (39) for computing $\partial_t Z$, we get the following equation:

$$(41) \quad \partial_t W + B_{22}(D)W = -(B_{22}^{-1}A_{21})(D)(A_{11}(D)Z_1 + A_{12}(D)Z_2) - (B_{22}^{-1}A_{22}B_{22})(D)W.$$

Rewriting the equation of Z_1 in terms of W yields

$$(42) \quad \partial_t Z_1 + (A_{11}(D) - (A_{12}B_{22}^{-1}A_{21})(D))Z_1 = (A_{12}B_{22}^{-1}A_{22})(D)Z_2 - A_{12}(D)W.$$

³That are in particular satisfied by the linearized Euler equations with relaxation.

In order to pursue our analysis, we make the following assumption:

$$(43) \quad A_{11}(D) \equiv 0 \quad \text{and} \quad \mathcal{A}(D) \triangleq -(A_{12}B_{22}^{-1}A_{21})(D) \quad \text{is a positive operator.}$$

By positive, we mean that the symbol $A_{12}B_{22}^{-1}A_{21}$ has range in the set of positive Hermitian matrices of size n_2 . For this particular structure, the above hypothesis turns out to be equivalent to Condition (SK) (see Lemma A.3).

Then, after applying $\dot{\Delta}_j$ to (41) and for (42), we obtain:

$$(44) \quad \begin{cases} \partial_t Z_{1,j} + \mathcal{A}(D)Z_{1,j} = (A_{12}B_{22}^{-1}A_{22})(D)Z_{2,j} - A_{12}(D)W_j, \\ \partial_t W_j + B_{22}(D)W_j = -(B_{22}^{-1}A_{21}A_{12})(D)Z_{2,j} - (B_{22}^{-1}A_{22}B_{22})(D)W_j. \end{cases}$$

Using Duhamel formula for computing $Z_{1,j}$ from the first equation of (44), we get

$$Z_{1,j}(t) = e^{-t\mathcal{A}(D)}Z_{1,j}(0) + \int_0^t e^{-(t-\tau)\mathcal{A}(D)}((A_{12}B_{22}^{-1}A_{22})(D)Z_{2,j}(\tau) - A_{12}(D)W_j(\tau))d\tau.$$

Since $\mathcal{A}(D)$ is second order positive and satisfies the assumptions of Lemma A.2, there exist two constants c and C such that the following bound holds:

$$(45) \quad \|e^{-t\mathcal{A}(D)}\dot{\Delta}_j z\|_{L^p(\mathbb{R}^{n_1};\mathbb{R}^{n_1})} \leq Ce^{-c2^{2j}t}\|\dot{\Delta}_j z\|_{L^p(\mathbb{R}^{n_1};\mathbb{R}^{n_1})}, \quad j \in \mathbb{Z}.$$

Then, we get from Bernstein inequality (151), remembering that all the blocks of $A(D)$ are homogeneous multipliers of degree 1 and that $B_{22}^{-1}(D)$ is homogeneous of degree 0,

$$\|Z_{1,j}(t)\|_{L^p} \lesssim e^{-c2^{2j}t}\|Z_{1,j}(0)\|_{L^p} + \int_0^t e^{-c2^{2j}(t-\tau)}(2^{2j}\|Z_{2,j}(\tau)\|_{L^p} + 2^j\|W_j(\tau)\|_{L^p})d\tau,$$

whence taking the supremum or the integral on $[0, t]$,

$$\|Z_{1,j}(t)\|_{L^p} + 2^{2j} \int_0^t \|Z_{1,j}\|_{L^p} d\tau \lesssim \|Z_{1,j}(0)\|_{L^p} + \int_0^t (2^{2j}\|Z_{2,j}\|_{L^p} + 2^j\|W_j\|_{L^p}) d\tau.$$

Similarly, Lemma A.3 guarantees that we have

$$(46) \quad \|e^{-tB_{22}(D)}\dot{\Delta}_j z\|_{L^p(\mathbb{R}^{n_2};\mathbb{R}^{n_2})} \leq Ce^{-ct}\|\dot{\Delta}_j z\|_{L^p(\mathbb{R}^{n_2};\mathbb{R}^{n_2})}, \quad j \in \mathbb{Z},$$

which allows to get eventually

$$\|W_j(t)\|_{L^p} + \int_0^t \|W_j\|_{L^p} d\tau \lesssim \|W_j(0)\|_{L^p} + 2^{2j} \int_0^t \|Z_j\|_{L^p} d\tau + 2^j \int_0^t \|W_j\|_{L^p} d\tau.$$

Owing to the factor 2^j , there exists an integer $j_0 \in \mathbb{Z}$ so that the last term may be absorbed by the left-hand side for all $j \leq j_0$. Hence, multiplying by 2^{j_0} then summing up on $j \leq j_0$ yields, with the notation $\|z\|_{\dot{B}_{p,1}^{s,j_0}} \triangleq \sum_{j \leq j_0} 2^{j_0} \|\dot{\Delta}_j z\|_{L^p}$,

$$(47) \quad \|W(t)\|_{\dot{B}_{p,1}^{s,j_0}} + \int_0^t \|W\|_{\dot{B}_{p,1}^{s,j_0}} d\tau \lesssim \|W_0\|_{\dot{B}_{p,1}^{s,j_0}} + \int_0^t \|Z\|_{\dot{B}_{p,1}^{s+2,j_0}} d\tau$$

while the inequality for Z_1 gives us

$$(48) \quad \|Z_1(t)\|_{\dot{B}_{p,1}^{s,j_0}} + \int_0^t \|Z_1\|_{\dot{B}_{p,1}^{s+2,j_0}} d\tau \lesssim \|Z_{1,0}\|_{\dot{B}_{p,1}^{s,j_0}} + \int_0^t (\|Z_2\|_{\dot{B}_{p,1}^{s+2,j_0}} + \|W\|_{\dot{B}_{p,1}^{s+1,j_0}}) d\tau.$$

The definition of W in (40) ensures that for all $j \leq j_0$ (with negative enough j_0), there holds that

$$(49) \quad \|W_j\|_{L^p} \lesssim \|Z_{2,j}\|_{L^p} + 2^j\|Z_{1,j}\|_{L^p} \quad \text{and} \quad \|Z_{2,j}\|_{L^p} \lesssim \|W_j\|_{L^p} + 2^j\|Z_{1,j}\|_{L^p}.$$

Hence, adding up $\varepsilon \cdot (47)$ to (48) with ε small enough and negative enough j_0 , we conclude that

$$\|Z(t)\|_{\dot{B}_{p,1}^{s,j_0}}^{\ell,j_0} + \int_0^t (\|Z_1\|_{\dot{B}_{p,1}^{s+2}}^{\ell,j_0} + \|W\|_{\dot{B}_{p,1}^s}^{\ell,j_0}) d\tau \lesssim \|Z_0\|_{\dot{B}_{p,1}^s}^{\ell,j_0}.$$

Of course, combining with (49) yields also

$$\int_0^t \|Z_2\|_{\dot{B}_{p,1}^{s+1}}^{\ell,j_0} d\tau \lesssim \|Z_0\|_{\dot{B}_{p,1}^s}^{\ell,j_0}.$$

By the same token, if we consider a source term F in (39), one gets the following bound:

$$\|Z(t)\|_{\dot{B}_{p,1}^s}^{\ell,j_0} + \int_0^t (\|Z_1\|_{\dot{B}_{p,1}^{s+2}}^{\ell,j_0} + \|Z_2\|_{\dot{B}_{p,1}^{s+1}}^{\ell,j_0} + \|W\|_{\dot{B}_{p,1}^s}^{\ell,j_0}) d\tau \lesssim \|Z_0\|_{\dot{B}_{p,1}^s}^{\ell,j_0} + \int_0^t \|F\|_{\dot{B}_{p,1}^s}^{\ell,j_0} d\tau,$$

which is actually the same as the one we proved before for $p = 2$.

At the linear level, there is no restriction on the value of p : it can be any element of $[1, \infty]$. Reverting to the initial nonlinear system (11), it is possible to work out a functional framework of L^p type for the low frequencies of the solution. However, owing to the interactions between the low and high frequencies through the nonlinear terms, there are some restrictions on p . The most obvious one is that, if combining Bernstein and Hölder inequalities for estimating the medium frequencies in a L^2 type space of a product of low frequencies that belong to a L^p type space, one needs to have $p \in [2, 4]$. In high dimension, there are stronger restrictions on p . The reader is referred to [10, 12] for more details and complete statements.

2. GLOBAL EXISTENCE IN THE CRITICAL REGULARITY SETTING

The principal aim of this section is to prove the global existence of strong solutions for (9) supplemented with initial data that are a perturbation of a constant state \bar{V} satisfying Condition (SK). For notational simplicity, we assume that $\bar{V} = 0$ so that the system under consideration reads⁴

$$(50) \quad \partial_t Z + \sum_{k=1}^d A^k(Z) \cdot \partial_k Z + BZ = 0.$$

It is assumed that the (smooth) given functions A^1, \dots, A^d range in the set of $n \times n$ real symmetric matrices, and that $B = \begin{pmatrix} 0 & 0 \\ 0 & L_2 \end{pmatrix}$ with $L_2 \in GL_{n_2}(\mathbb{R})$ satisfying for some $c > 0$,

$$(51) \quad L_2 z \cdot z \geq c|z|^2, \quad z \in \mathbb{R}^{n_2}.$$

Set $A(\xi) \triangleq i \sum_{k=1}^d \xi_k \bar{A}^k$ with $\bar{A}^k \triangleq A^k(0)$, and $B(\xi) \triangleq B$. According to the linear analysis that was performed in the previous paragraph in the context of System (50), Condition (SK) is equivalent to:

$$(52) \quad \text{Rank} \begin{pmatrix} B(\xi) \\ BA(\xi) \\ \vdots \\ BA^{n-1}(\xi) \end{pmatrix} = n \quad \text{for all } \xi \in \mathbb{R}^d \setminus \{0\}.$$

⁴The reader is referred to [11] for the proof of similar results for more general symmetrizable quasilinear partially dissipative hyperbolic systems satisfying (SK).

2.1. The main results. In order to find out a suitable functional framework for solving (50), let us temporarily consider a smooth solution Z . Taking advantage of the symmetry of the matrices A^k and integrating by parts, one gets the following ‘energy identity’:

$$\frac{1}{2} \frac{d}{dt} \|Z\|_{L^2}^2 - \frac{1}{2} \sum_{k,\ell,m} \int_{\mathbb{R}^d} Z^\ell Z^m \partial_k (A_{\ell m}^k(Z)) dx + \int_{\mathbb{R}^d} BZ \cdot Z dx = 0.$$

Therefore, combining with (51) and Gronwall inequality, we discover that

$$\|Z(t)\|_{L^2}^2 + c \int_0^t \|Z_2\|_{L^2}^2 d\tau \leq \|Z_0\|_{L^2}^2 \exp\left(C \int_0^t \|\nabla Z\|_{L^\infty} d\tau\right).$$

Hence, even for controlling the L^2 norm of the solution, a bound of ∇Z in $L^1_{loc}(\mathbb{R}_+; L^\infty)$ is needed. Since no gain of regularity can be expected on the whole solution (see (34)), we must assume that Z_0 belongs to a functional space X that is embedded in the set of globally Lipschitz functions. If $X = H^s$ then this embedding holds if and only if $s > d/2 + 1$. In the framework of Besov spaces with last index 1, one can reach the critical index $s = d/2 + 1$, owing to the (critical) embedding

$$(53) \quad \dot{B}_{2,1}^{\frac{d}{2}}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d).$$

Hence, X must be a subspace of $\dot{B}_{2,1}^{\frac{d}{2}+1}$. Consequently, we shall take $s' = 1 + d/2$ in (34).

As regards the value of the regularity exponent s in (33) for the low frequencies, a natural candidate is $s = -1 + d/2$ since (33) and (34) together give us a control of Z in $L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})$ (provided we succeed in bounding in $L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})$ the nonlinear term F), and thus of ∇Z in $L^1(\mathbb{R}_+; L^\infty)$. Having at our disposal global L^1 -in-time estimates for the solution will be particularly comfortable for further analysis in contrast with the classical ‘Sobolev’ approaches for partially dissipative systems where only L^2 -in time estimates are available.

To make a long story short, a good candidate for a solution space is the set of functions Z in $C_b^1(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^n)$ such that

$$Z^\ell \in C_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1}) \cap L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1}) \quad \text{and} \quad Z^h \in C_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1}) \cap L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1}).$$

According to the linear analysis presented before, one can expect to get additional informations for low frequencies, through the damped mode W defined in (40), that is essentially equivalent to $\partial_t Z_2$ in our context. We will eventually obtain the following result that will be proved in the next subsection.

Theorem 2.1. *Let the Conditions (51) and (52) be in force and assume that $d \geq 2$. Then, there exists a positive constant α such that for all $Z_0 \in \widetilde{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}$ satisfying*

$$(54) \quad Z_0 \triangleq \|Z_0\|_{\widetilde{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}} \leq \alpha,$$

System (50) supplemented with initial data Z_0 admits a unique global-in-time solution Z in the set

$$E \triangleq \left\{ Z \in C_b(\mathbb{R}_+; \widetilde{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}), Z \in L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1}) \quad \text{and} \quad \partial_t Z_2 \in L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1}) \right\}.$$

Moreover, there exist an explicit Lyapunov functional, equivalent to $\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}}$ and a constant C depending only on the matrices A^k and on L_2 , and such that

$$(55) \quad \mathcal{Z}(t) \leq C \mathcal{Z}_0 \quad \text{for all } t \geq 0$$

where⁵

$$(56) \quad \mathcal{Z}(t) \triangleq \|Z\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})}^\ell + \|Z\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|Z\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \\ + \|\partial_t Z_2\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|Z_2\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|Z_2\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}-1})}^\ell.$$

Choosing regularity $d/2 - 1$ for low frequencies has some disadvantages, though:

- it does not allow to treat the mono-dimensional case since the low frequencies of the nonlinear terms of type $DZ \times Z$ cannot be estimated in $L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})$ (this is the needed regularity for the right-hand side of (33)). Indeed, the numerical product does not map $\dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}) \times \dot{B}_{2,1}^{-\frac{1}{2}}(\mathbb{R})$ to $\dot{B}_{2,1}^{-\frac{1}{2}}(\mathbb{R})$.
- it does not provide us with uniform bounds in the high relaxation asymptotics (see the beginning of Section 4 for more explanations).

Another possible choice is $s = d/2$. Then, the solution space becomes the set of Z in $\mathcal{C}_b^1(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^n)$ satisfying

$$Z^\ell \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}) \cap L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+2}) \quad \text{and} \quad Z^h \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1}) \cap L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})$$

plus crucial informations from the damped mode that, in particular, will ensure that

$$\nabla Z_2 \in L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}) \quad \text{and} \quad Z_2 \in L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}).$$

This alternative framework allows to consider initial data that are less decaying at infinity (regularity $\dot{B}_{2,1}^{\frac{d}{2}}$ for low frequencies is less stringent than $\dot{B}_{2,1}^{\frac{d}{2}-1}$), to handle the one-dimensional situation, and to provide crucial uniform a priori bounds in the strong relaxation limit. The only drawback is that this alternative framework requires seemingly stronger structure assumptions on the system (that are nevertheless fulfilled by the compressible Euler equations). In order to specify them, let us rewrite System (50) by blocks as follows:

$$(57) \quad \begin{cases} \partial_t Z_1 + \sum_{k=1}^d \left(A_{11}^k(Z) \partial_k Z_1 + A_{12}^k(Z) \partial_k Z_2 \right) = 0, \\ \partial_t Z_2 + \sum_{k=1}^d \left(A_{21}^k(Z) \partial_k Z_1 + A_{22}^k(Z) \partial_k Z_2 \right) + L_2 Z_2 = 0. \end{cases}$$

Then, we need the following additional assumption:

(H3) For all $k \in \{1, \dots, d\}$, $\bar{A}_{11}^k = 0$ and $Z \mapsto A_{11}^k(Z)$ is linear with respect to Z_2 .

Note that in the context of gas dynamics, the above assumption just means that there are no terms like $\nabla \varrho$ or $\varrho \nabla \varrho$ in the density equation, which is indeed the case !

⁵Whenever X is a Banach space, $p \in [1, \infty]$ and $T \geq 0$, notation $\|\cdot\|_{L_T^p(X)}$ designates the Lebesgue norm L^p of functions on $[0, T]$ with values in X .

Theorem 2.2. *In general dimension $d \geq 1$, let the assumptions of Theorem 2.1 concerning system (9) be in force and assume in addition that **(H3)** holds true. Then, there exists a positive constant α such that for all $Z_0 \in \widetilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}$ satisfying*

$$(58) \quad \mathcal{Z}'_0 \triangleq \|Z_0\|_{\widetilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}} \leq \alpha,$$

System (57) supplemented with initial data Z_0 admits a unique global-in-time solution in the subspace F of functions Z of $\mathcal{C}_b(\mathbb{R}_+; \widetilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1})$ such that

$$Z_2^\ell, Z^h \in L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1}), \quad Z_1^\ell \in L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+2}) \quad \text{and} \quad \partial_t Z_2 \in L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}).$$

Moreover, there exists an explicit Lyapunov functional that is equivalent to $\|Z\|_{\widetilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}}$ and we have the following bound:

$$(59) \quad \mathcal{Z}'(t) \leq C \mathcal{Z}'_0 \quad \text{where} \quad \mathcal{Z}'(t) \triangleq \|Z\|_{L_t^\infty(\widetilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1})} + \|Z_1\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell \\ + \|Z_2\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \|Z_2\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|Z\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|\partial_t Z_2\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}.$$

Theorem 2.2 directly applies to the isentropic compressible Euler equations with relaxation, written in terms of the sound speed c and of v (that is, System (5)). The result we get reads as follows:

Theorem 2.3. *Let $\bar{c} > 0$, $d \geq 1$ and $\gamma > 1$. There exists a positive constant α such that for any data (c_0, v_0) such that $c_0 - \bar{c}$ and v_0 belong to $\widetilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}$, and satisfy*

$$(60) \quad \mathcal{A}_0 \triangleq \|(c_0 - \bar{c}, v_0)\|_{\widetilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}} \leq \alpha,$$

System (5) with $\varepsilon = 1$ admits a unique global solution (c, v) with $(c - \bar{c}, v) \in \mathcal{C}_b(\mathbb{R}_+; \widetilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1})$ that satisfies

$$(61) \quad \|(c - \bar{c}, v)\|_{L^\infty(\mathbb{R}_+; \widetilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1})} + \|c - \bar{c}\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+2})}^\ell \\ + \|v\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \|v\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})}^\ell + \|(c - \bar{c}, v)\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|\partial_t v\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \leq C \mathcal{A}_0.$$

A similar statement holds true for the barotropic compressible Euler equations with general smooth pressure law P satisfying $P' > 0$ in the neighborhood of the reference density $\bar{\rho}$, although one cannot ‘symmetrize’ the system any longer by using the sound speed. For more details, the reader may refer to [11] where a class of partially dissipative systems, more general than (9), is considered.

In the rest of this section, we focus on the proof of Theorem 2.1. The reader is referred to [11] for more general systems and for the proof of Theorem 2.2. A similar statement in the L^p framework has been established in [12].

2.2. A priori estimates. The overall strategy is to apply the Littlewood-Paley truncation operator $\dot{\Delta}_j$ to (50), then to follow the method that has been described in the previous section so as to get optimal estimates in L^2 for each dyadic block. Performing eventually a suitable weighted summation on j will lead to the control of Besov norms of the solution, as stated in Theorem 2.1.

Throughout, we assume that we are given a smooth and sufficiently decaying solution of (50) on $[0, T] \times \mathbb{R}^d$ such that

$$(62) \quad \sup_{t \in [0, T]} \|z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}(\mathbb{R}^d)} \text{ is sufficiently small.}$$

We shall use repeatedly that, owing to the embedding (53), the solution Z is also small in $L^\infty([0, T] \times \mathbb{R}^d)$.

• *Low frequencies.* Let us denote $Z_j \triangleq \dot{\Delta}_j Z$ and $F_j \triangleq \dot{\Delta}_j F$, with

$$(63) \quad F \triangleq \sum_{k=1}^d (\bar{A}^k - A^k(Z)) \partial_k Z.$$

We see that for all $j \in \mathbb{Z}$,

$$(64) \quad \partial_t Z_j + \sum_{k=1}^d \bar{A}^k \partial_k Z_j + B Z_j = F_j.$$

Hence, taking the $L^2(\mathbb{R}^d; \mathbb{R}^n)$ scalar product with Z_j and using that the first order terms are skew-symmetric yields

$$\frac{1}{2} \frac{d}{dt} \|Z_j\|_{L^2}^2 + \int_{\mathbb{R}^d} B Z_j \cdot Z_j \, dx = \int_{\mathbb{R}^d} Z_j \cdot F_j \, dx.$$

The term with B may be bounded from below according to (51). Hence, using Cauchy-Schwarz inequality for bounding the right-hand side delivers for some $c > 0$,

$$\frac{1}{2} \frac{d}{dt} \|Z_j\|_{L^2}^2 + c \|Z_{2,j}\|_{L^2}^2 \leq \|F_j\|_{L^2} \|Z_j\|_{L^2}.$$

In order to recover the full dissipation, we proceed as in the previous section, introducing the functional $L_{r,\omega}$ defined in (26). Adapting the computations therein to the case where the source term in (12) is nonzero, we get for all $r > 0$ and $\omega \in \mathbb{S}^{d-1}$ (with the notations of (20)):

$$\begin{aligned} \frac{d}{dt} L_{r,\omega}(\widehat{Z}_j) + \frac{\min(1, r^2)}{2} \sum_{k=0}^{n-1} \varepsilon_k |B_\omega A_\omega^k \widehat{Z}_j|^2 &\leq \operatorname{Re}(\widehat{F}_j \cdot \widehat{Z}_j) \\ &+ \min(r, r^{-1}) \sum_{k=1}^{n-1} \left(\operatorname{Re}(B_\omega A_\omega^{k-1} \widehat{Z}_j \cdot B_\omega A_\omega^k \widehat{F}_j) + \operatorname{Re}(B_\omega A_\omega^{k-1} \widehat{F}_j \cdot B_\omega A_\omega^k \widehat{Z}_j) \right). \end{aligned}$$

In light of Cauchy-Schwarz inequality, the sum in the right-hand side may be bounded by $\sqrt{L_{r,\omega}(\widehat{Z}_j) L_{r,\omega}(\widehat{F}_j)}$. Hence, using that Condition (SK) and (27) ensure the existence of a positive constant $c_0 > 0$ such that for all $r > 0$ and $\omega \in \mathbb{S}^{d-1}$,

$$(65) \quad \sum_{k=0}^{n-1} \varepsilon_k |B_\omega A_\omega^k \widehat{Z}_j|^2 \geq 2c_0 L_{r,\omega}(\widehat{Z}_j),$$

we conclude that

$$(66) \quad \frac{d}{dt} L_{r,\omega}(\widehat{Z}_j) + c_0 \min(1, r^2) L_{r,\omega}(\widehat{Z}_j) \leq \sqrt{L_{r,\omega}(\widehat{Z}_j) L_{r,\omega}(\widehat{F}_j)}.$$

Let us denote for all $j \in \mathbb{Z}$,

$$\mathcal{L}_j \triangleq \|\widehat{Z}_j\|_{L^2}^2 + \sum_{k=1}^{n-1} \varepsilon_k \operatorname{Re} \int_{\mathbb{R}^d} (BA_\omega^{k-1} \widehat{Z}_j(\xi)) \cdot (BA_\omega^k \widehat{Z}_j(\xi)) \min(|\xi|, |\xi|^{-1}) d\xi.$$

Integrating (66) on \mathbb{R}^d and observing that, by virtue of (27), we have

$$(67) \quad \frac{1}{2} \|Z_j\|_{L^2}^2 \leq \mathcal{L}_j \leq 2 \|Z_j\|_{L^2}^2,$$

we get, up to a slight modification of c_0 ,

$$(68) \quad \frac{d}{dt} \mathcal{L}_j + c_0 \min(1, 2^{2j}) \mathcal{L}_j \leq C \|F_j\|_{L^2} \sqrt{\mathcal{L}_j}.$$

Therefore, taking $X = \sqrt{\mathcal{L}_j}$ and $A = C \|F_j\|_{L^2}$ in Lemma A.1, then multiplying by $2^{j(\frac{d}{2}-1)}$ delivers for all $j < 0$,

$$2^{j(\frac{d}{2}-1)} \sqrt{\mathcal{L}_j(t)} + \frac{c_0}{2} 2^{j(\frac{d}{2}+1)} \int_0^t \sqrt{\mathcal{L}_j} d\tau \leq 2^{j(\frac{d}{2}-1)} \sqrt{\mathcal{L}_j(0)} + C 2^{j(\frac{d}{2}-1)} \int_0^t \|F_j\|_{L^2} d\tau.$$

In order to bound the right-hand side, it suffices to combine the following facts that are proved in e.g. [2, Chap. 2]:

$$(69) \quad \text{For } d \geq 2, \text{ the numerical product maps } \dot{B}_{2,1}^{\frac{d}{2}-1}(\mathbb{R}^d) \times \dot{B}_{2,1}^{\frac{d}{2}}(\mathbb{R}^d) \text{ to } \dot{B}_{2,1}^{\frac{d}{2}-1}(\mathbb{R}^d)$$

and, for all smooth function $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^p$ vanishing at 0,

$$(70) \quad \|\Phi(Z)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq C (\|Z\|_{L^\infty}) \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

Hence, remembering also (62), we conclude that

$$\|F\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \lesssim \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

So, finally, there exist two positive constants c_0 and C such that for all $j < 0$, we have

$$(71) \quad 2^{j(\frac{d}{2}-1)} \sqrt{\mathcal{L}_j(t)} + c_0 2^{j(\frac{d}{2}+1)} \int_0^t \sqrt{\mathcal{L}_j} d\tau \leq 2^{j(\frac{d}{2}-1)} \sqrt{\mathcal{L}_j(0)} \\ + C \int_0^t c_j \|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} d\tau \quad \text{with} \quad \sum_{j \in \mathbb{Z}} c_j = 1.$$

• *High frequencies.* In order to bound the high frequency part of the solution, we shall keep the functional \mathcal{L}_j , but one cannot look at F defined in (63) as a source term since this would entail a loss of one derivative. To overcome the difficulty, we mimic the proof of the L^2 estimate recalled at the beginning of this section, writing the system for $Z_j \triangleq \dot{\Delta}_j Z$ as follows:

$$(72) \quad \partial_t Z_j + \sum_{k=1}^d A^k(Z) \partial_k Z_j + B Z_j = \sum_{k=1}^d [A^k(Z), \dot{\Delta}_j] \partial_k Z.$$

Then, taking the $L^2(\mathbb{R}^d; \mathbb{R}^n)$ scalar product with Z_j and integrating by parts yields:

$$\frac{1}{2} \frac{d}{dt} \|Z_j\|_{L^2}^2 + \int_{\mathbb{R}^d} B Z_j \cdot Z_j dx = \frac{1}{2} \sum_{k,\ell,m} \int_{\mathbb{R}^d} \partial_k (A_{\ell m}^k(Z)) Z_j^\ell Z_j^m dx + \sum_{k=1}^d \int_{\mathbb{R}^d} [A^k(Z), \dot{\Delta}_j] Z \cdot Z_j dx.$$

The last sum may be bounded according to the following classical commutator estimate (see e.g. [2, Chap. 2]) that is valid for all $s \in (-d/2, d/2 + 1]$:

$$\|[A^k(Z), \dot{\Delta}_j] \partial_k Z\|_{L^2} \leq C c_j 2^{-js} \|\nabla(A^k(Z))\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|Z\|_{\dot{B}_{2,1}^s} \quad \text{with} \quad \sum_{j \in \mathbb{Z}} c_j = 1.$$

Thanks to the embedding (53) and to the definition of $\|\cdot\|_{\dot{B}_{2,1}^s}$, we have

$$\sum_{k,\ell,m} \int_{\mathbb{R}^d} \partial_k(A_{\ell m}^k(Z)) Z_j^\ell Z_j^m dx \leq C c_j 2^{-js} \|\nabla(A^k(Z))\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|Z\|_{\dot{B}_{2,1}^s} \|Z_j\|_{L^2}.$$

Hence, owing to (51), we have for all $s \in (-d/2, d/2 + 1]$,

$$(73) \quad \frac{1}{2} \frac{d}{dt} \|Z_j\|_{L^2}^2 + c \|Z_{2,j}\|_{L^2}^2 \leq C c_j 2^{-js} \|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|Z\|_{\dot{B}_{2,1}^s} \|Z_j\|_{L^2}.$$

To recover the full dissipation, one has to compute for all $r \geq 1$ and $\omega \in \mathbb{S}^{d-1}$, the time derivative of

$$r^{-1} \tilde{L}_{r,\omega}(\widehat{Z}_j) \quad \text{with} \quad \tilde{L}_{r,\omega}(\widehat{Z}_j) \triangleq \sum_{k=1}^{n-1} \varepsilon_k \operatorname{Re}(BA_\omega^{k-1} \widehat{Z}_j \cdot BA_\omega^k)$$

as it will generate the term $\sum_{k=1}^{n-1} \frac{\varepsilon_k}{2} |BA_\omega^k \widehat{Z}_j|^2$, that is, the missing dissipation. To proceed, one can keep F defined in (63) as a source term and start from (64). For $j \geq 0$, the term r^{-1} yields the factor 2^{-j} that exactly compensates the loss of one derivative when estimating F in $\dot{B}_{2,1}^{\frac{d}{2}+1}$. Hence, it suffices to estimate F in $\dot{B}_{2,1}^{\frac{d}{2}}$, which may be done by combining (70) and the following fact:

$$(74) \quad \text{the numerical product maps } \dot{B}_{2,1}^{\frac{d}{2}}(\mathbb{R}^d) \times \dot{B}_{2,1}^{\frac{d}{2}}(\mathbb{R}^d) \text{ to } \dot{B}_{2,1}^{\frac{d}{2}}(\mathbb{R}^d).$$

Remembering (62), we get

$$(75) \quad \|F\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

Now, adding up the relation we get for $r^{-1} \tilde{L}_{r,\omega}(\widehat{Z}_j)$ (after space integration) to (73) yields for all $j \geq 0$:

$$(76) \quad \frac{d}{dt} \mathcal{L}_j + \frac{1}{2} \sum_{k=0}^{n-1} \int_{\mathbb{R}^d} \varepsilon_k |BA_\omega^k Z_j(\xi)|^2 d\xi \lesssim (2^{-j} \|F_j\|_{L^2} + c_j 2^{-j(\frac{d}{2}+1)} \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^2) \|Z_j\|_{L^2},$$

and (65) guarantees that

$$(77) \quad \sum_{k=0}^{n-1} \int_{\mathbb{R}^d} \varepsilon_k |BA_\omega^k Z_j(\xi)|^2 d\xi \gtrsim \mathcal{L}_j.$$

Hence, using Lemma A.1, multiplying by $2^{j(\frac{d}{2}+1)}$ and taking advantage of (63) and (75), we end up with

$$(78) \quad 2^{j(\frac{d}{2}+1)} \sqrt{\mathcal{L}_j(t)} + c 2^{j(\frac{d}{2}+1)} \int_0^t \sqrt{\mathcal{L}_j} d\tau \\ \leq 2^{j(\frac{d}{2}+1)} \sqrt{\mathcal{L}_j(0)} + C \int_0^t c_j \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} d\tau.$$

• *Conclusion.* Let us put

$$(79) \quad \mathcal{L} \triangleq \sum_{j<0} 2^{j(\frac{d}{2}-1)} \sqrt{\mathcal{L}_j} + \sum_{j \geq 0} 2^{j(\frac{d}{2}+1)} \sqrt{\mathcal{L}_j} \quad \text{and} \quad \mathcal{H} \triangleq \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}.$$

Since $\mathcal{L}_j \simeq \|Z_j\|_{L^2}^2$, we have the following equivalence:

$$(80) \quad \mathcal{L} \simeq \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}}.$$

Note that this implies that

$$(81) \quad \|Z\|_{\dot{B}_{2,1}^s} \lesssim \mathcal{L} \quad \text{for all} \quad \frac{d}{2} - 1 \leq s \leq \frac{d}{2} + 1.$$

Hence, we deduce from (71) and (78) that

$$(82) \quad \mathcal{L}(t) + c \int_0^t \mathcal{H}(\tau) d\tau \leq \mathcal{L}(0) + C \int_0^t \mathcal{H}(\tau) \mathcal{L}(\tau) d\tau.$$

We claim that there exists $\alpha > 0$ such that if $\mathcal{L}(0) < \alpha$ then, for all $t \in [0, T]$, we have

$$(83) \quad \mathcal{L}(t) + \frac{c}{2} \int_0^t \mathcal{H}(\tau) d\tau \leq \mathcal{L}(0).$$

Indeed, let us choose $\alpha \in (0, c/(2C))$ so that $\mathcal{L} \leq \alpha$ implies that (62) is satisfied, and set

$$T_0 \triangleq \sup\{T_1 \in [0, T], \sup_{t \in [0, T_1]} \mathcal{L}(t) \leq \alpha\}.$$

The above set is nonempty (as 0 is in it) and contains its supremum since \mathcal{L} is continuous (remember that we assumed that Z is smooth). Hence we have

$$\mathcal{L}(T_0) + c \int_0^{T_0} \mathcal{H}(\tau) d\tau \leq \mathcal{L}(0) + C \int_0^{T_0} \mathcal{H}(\tau) \mathcal{L}(\tau) d\tau \leq \mathcal{L}(0) + \frac{c}{2} \int_0^{T_0} \mathcal{H}(\tau) d\tau.$$

Using the smallness hypothesis on $\mathcal{L}(0)$, one may conclude that $\mathcal{L} < \alpha$ on $[0, T_0]$. As \mathcal{L} is continuous, we must have $T_0 = T$ and (83) thus holds on $[0, T]$.

Clearly, time $t = 0$ does not play any particular role, and one can apply the same argument on any sub-interval of $[0, T]$, which leads to:

$$(84) \quad \mathcal{L}(t) + \frac{c}{2} \int_{t_0}^t \mathcal{H}(\tau) d\tau \leq \mathcal{L}(t_0), \quad 0 \leq t_0 \leq t \leq T.$$

Hence, provided that $\|Z_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}}$ is small enough, \mathcal{L} is a Lyapunov functional that is, in light of (80), equivalent to $\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}}$.

2.3. The damped mode. Define W by the relation:

$$\partial_t Z_2 + L_2 W = 0.$$

Since L_2 is invertible, the second line of (57) yields

$$(85) \quad W = Z_2 + L_2^{-1} \sum_{k=1}^d (A_{21}^k(Z) \partial_k Z_1 + A_{22}^k(Z) \partial_k Z_2),$$

which allows to get the following equation for W :

$$(86) \quad \partial_t W + L_2 W = L_2^{-1} \left(\sum_{k=1}^d \partial_t (A_{21}^k(Z) \partial_k Z_1) + \partial_t (A_{22}^k(Z)) \partial_k Z_2 - A_{22}^k(Z) L_2 \partial_k W \right).$$

Applying $\dot{\Delta}_j$ to the above relation and denoting $W_j \triangleq \dot{\Delta}_j W$ leads to

$$\begin{aligned} \partial_t W_j + L_2 W_j = L_2^{-1} & \left(\sum_{k=1}^d \dot{\Delta}_j \partial_t ((A_{21}^k(Z) - \bar{A}_{21}^k) \partial_k Z_1) + \bar{A}_{21}^k \partial_t \partial_k Z_{1,j} \right. \\ & \left. + \dot{\Delta}_j (\partial_t (A_{22}^k(Z)) \partial_k Z_2) + \dot{\Delta}_j ((\bar{A}_{22}^k - A_{22}^k(Z)) W) - \bar{A}_{22}^k L_2 \partial_k W_j \right). \end{aligned}$$

Using (51), an energy method and Lemma A.1, we get two positive constants c and C such that for all $j \in \mathbb{Z}$ and $t \in [0, T]$,

$$(87) \quad \begin{aligned} \|W_j(t)\|_{L^2} + c \int_0^t \|W_j\|_{L^2} & \leq \|W_j(0)\|_{L^2} \\ & + C \int_0^t \sum_{k=1}^d \left(\|\dot{\Delta}_j \partial_t ((A_{21}^k(Z) - \bar{A}_{21}^k) \partial_k Z_1)\|_{L^2} + \|\bar{A}_{21}^k \partial_t \partial_k Z_{1,j}\|_{L^2} \right. \\ & \left. + \|\dot{\Delta}_j (\partial_t (A_{22}^k(Z)) \partial_k Z_2)\|_{L^2} + \|\dot{\Delta}_j ((\bar{A}_{22}^k - A_{22}^k(Z)) W)\|_{L^2} + \|L_2^{-1} \bar{A}_{22}^k L_2 \partial_k W_j\|_{L^2} \right) d\tau. \end{aligned}$$

Bernstein inequality (150) guarantees that

$$\|L_2^{-1} \bar{A}_{22}^k L_2 \partial_k W_j\|_{L^2} \leq C 2^j \|W_j\|_{L^2}.$$

Hence, there exists $j_0 \in \mathbb{Z}$ such that for all $j \leq j_0$, the last term may be absorbed by the time integral of the left-hand side.

Next, using (57) to compute the time derivatives, we see that the terms with

$$\partial_t ((A_{21}^k(Z) - \bar{A}_{21}^k) \partial_k Z_1) \quad \text{or} \quad \partial_t (A_{22}^k(Z)) \partial_k Z_2$$

are linear combinations of coefficients of type $K(Z) Z \otimes \nabla^2 Z$, $K(Z) Z \otimes \nabla Z$ and $K(Z) \nabla Z \otimes \nabla Z$ for suitable smooth functions K . Hence, using (69), (70) and remembering (62) yields

$$\|\partial_t ((A_{21}^k(Z) - \bar{A}_{21}^k) \partial_k Z_1)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|\partial_t (A_{22}^k(Z)) \partial_k Z_2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \leq C \left(\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \right).$$

To handle $\|\dot{\Delta}_j ((\bar{A}_{22}^k - A_{22}^k(Z)) W)\|_{L^2}$, we split W into low and high frequencies. For the low frequency part, we just write that by composition (70) and product law (69),

$$\|\dot{\Delta}_j ((\bar{A}_{22}^k - A_{22}^k(Z)) W^\ell)\|_{L^2} \leq C c_j 2^{-j(\frac{d}{2}-1)} \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell.$$

For the high frequency part, we further decompose W as follows (in light of (85)):

$$(88) \quad \begin{aligned} W = Z_2 + L_2^{-1} \sum_{k=1}^d & (\bar{A}_{21}^k \partial_k Z_1 + \bar{A}_{22}^k \partial_k Z_2) \\ & + L_2^{-1} \sum_{k=1}^d ((A_{21}^k(Z) - \bar{A}_{21}^k) \partial_k Z_1 + (A_{22}^k(Z) - \bar{A}_{22}^k) \partial_k Z_2), \end{aligned}$$

which allows to get

$$(89) \quad \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \leq \|Z_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h + C\|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h + C\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}},$$

whence, using (74),

$$\begin{aligned} \|(\bar{A}_{22}^k - A_{22}^k(Z))W^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\|W^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \\ &\lesssim \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\left(\|Z_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h + \|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\right), \end{aligned}$$

and thus

$$\|\dot{\Delta}_j((\bar{A}_{22}^k - A_{22}^k(Z))W^h)\|_{L^2} \lesssim c_j 2^{-j\frac{d}{2}} \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} (1 + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \quad \text{with} \quad \sum_{j \in \mathbb{Z}} c_j = 1.$$

Plugging this information in (87), multiplying by $2^{j(\frac{d}{2}-1)}$, summing up on $j \leq j_0$ and remembering that $\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}$ is small, we conclude that⁶

$$\begin{aligned} \|W(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell + \frac{c}{2} \int_0^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell d\tau &\leq \|W_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell \\ &+ C \int_0^t \left(\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \right) d\tau + C \int_0^t \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^\ell d\tau. \end{aligned}$$

Since

$$\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h$$

and

$$\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \lesssim \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}},$$

taking advantage of (83) eventually yields :

$$(90) \quad \|W(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell + c \int_0^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell d\tau \leq C\mathcal{L}(0) \quad \text{for all } t \in [0, T].$$

Owing to (89) and (83), the high frequencies of W also satisfy

$$(91) \quad \|W(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^h + c \int_0^t \|W\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^h d\tau \leq C\mathcal{L}(0) \quad \text{for all } t \in [0, T],$$

which completes the proof of (55).

2.4. Proving Theorem 2.1. Having the a priori estimates (83), (90) and (91) at hand, constructing a global solution obeying Inequality (55) for any data Z_0 satisfying (54) follows from rather standard arguments. First, in order to benefit from the classical theory on first order hyperbolic systems, we remove the low frequency part of Z_0 so as to have an initial data in the *nonhomogeneous* Besov space $B_{2,1}^{\frac{d}{2}+1}$. More precisely, we set for all $n \in \mathbb{N}$,

$$(92) \quad Z_0^n \triangleq (\text{Id} - \dot{S}_n)Z_0 \quad \text{with} \quad \dot{S}_n \triangleq \chi(2^{-n}D).$$

⁶Handling the intermediate frequencies corresponding to $j_0 \leq j < 0$ may be done from (71) since, then, $2^{j(\frac{d}{2}+1)} \simeq 2^{j(\frac{d}{2}-1)}$ and $\|W_j\|_{L^2} \lesssim \|Z_j\|_{L^2}$.

In light of e.g. [2, Chap. 4], we get a unique maximal solution Z^n in

$$\mathcal{C}([0, T^n]; B_{2,1}^{\frac{d}{2}+1}) \cap \mathcal{C}^1([0, T^n]; B_{2,1}^{\frac{d}{2}}).$$

Since $B_{2,1}^{\frac{d}{2}}$ is embedded in $\dot{B}_{2,1}^{\frac{d}{2}}$, it is easy to prove from (57) and the composition and product laws (69) and (70) that $\partial_t Z_1$ and $(\partial_t Z_2 - L_2 Z_2)$ are in $L_{loc}^\infty(0, T^n; \dot{B}_{2,1}^{\frac{d}{2}-1})$, and as Z_0^n belongs to $\dot{B}_{2,1}^{\frac{d}{2}-1}$, we deduce that Z^n is actually in $\mathcal{C}([0, T^n]; B_{2,1}^{\frac{d}{2}+1} \cap B_{2,1}^{\frac{d}{2}-1})$, hence obeys (83) for all $t \in [0, T^n)$. In particular, the embedding $\dot{B}_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty$ guarantees that

$$\int_0^{T^n} \|\nabla Z^n\|_{L^\infty} dt < \infty,$$

and thus the standard continuation criterion for first order hyperbolic symmetric systems (again, refer to e.g. [2, Chap. 4]) ensures that $T^n = \infty$. In other words, for all $n \in \mathbb{N}$, the function Z^n is a global solution of (57) that satisfies (83), (90) and (91) for all $t \in \mathbb{R}_+$. Note that, owing to the definition (92), we have

$$\|Z_0^n\|_{\tilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}} \leq \|Z_0\|_{\tilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}}, \quad n \in \mathbb{N}.$$

Hence $(Z^n)_{n \in \mathbb{N}}$ is a sequence of global smooth solutions that is bounded in the space E of Theorem 2.1.

Proving the convergence of $(Z^n)_{n \in \mathbb{N}}$ relies on the following proposition that can be easily proved by writing out the system satisfied by the difference of two solutions Z and Z' of (57), namely,

$$\partial_t(Z - Z') + \sum_{k=1}^d A^k(Z) \cdot \partial_k(Z - Z') + B(Z - Z') = \sum_{k=1}^d (A^k(Z') - A^k(Z)) \partial_k Z',$$

applying the Littlewood-Paley truncation operator $\dot{\Delta}_j$ to this system then arguing as for getting (73) and using product laws (see the details in [11, Prop. 2]):

Proposition 2.1. *Consider two solutions Z and Z' of (57) in the space E corresponding to small enough initial data Z_0 and Z'_0 in $\tilde{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}$. Then we have for all $t \geq 0$,*

$$\|(Z - Z')(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq \|Z_0 - Z'_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + C \int_0^t (\|Z\|_{\tilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}} + \|Z'\|_{\tilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}}) d\tau.$$

From this proposition (applied to Z^n and Z^m for any $(n, m) \in \mathbb{N}^2$), Gronwall lemma and the definition of the initial data in (92), we gather that $(Z^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the space $\mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$, hence converges to some function Z in $\mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$. As the regularity is high, passing to the limit in the system is not an issue, and one can easily conclude that Z satisfies (57) supplemented with data Z_0 .

That Z belongs to the smaller space E stems from standard functional analysis. Typically, one uses that all the Besov spaces under consideration satisfy the Fatou property, that is, for instance

$$\|Z\|_{\tilde{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}} \leq C \liminf \|Z^n\|_{\tilde{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}}.$$

The only property that is missing is the time continuity with range in $\dot{B}_{2,1}^{\frac{d}{2}+1}$. However, this is known to be true for general quasilinear symmetric systems (see e.g. [2, Chap. 4]).

Finally, the uniqueness follows from Proposition 2.1.

3. DECAY ESTIMATES AND ASYMPTOTIC BEHAVIOR

The global-in-time properties of integrability for the solution Z that have been proved so far ensure that $Z(t)$ tends to 0 in the tempered distributional meaning when t goes to ∞ . In the present section, we aim at specifying the decay rate for some Besov norms of Z , whenever the initial data satisfy a (mild) additional condition. In the pioneering works by the Japanese school in the 70ies and early 80ies (see e.g. [28, 33]), it was expressed in terms of Lebesgue spaces L^p for some $p \in [1, 2)$. However, it is well understood now that it suffices to prescribe this condition in some homogeneous Besov spaces with a negative regularity index.

In order to understand how those spaces come into play, looking first at the linearized at the linearized system (12) with no source term is very informative. Let Z be the corresponding solution. Using (30) and Fourier-Plancherel theorem yields for all $t \geq 0$:

$$(93) \quad \|Z_j(t)\|_{L^2} \leq C \|Z_j(0)\|_{L^2} e^{-c \min(1, 2^{2j})t}, \quad j \in \mathbb{Z}.$$

This means that the high frequencies of Z decay to 0 exponentially fast, and that the low frequencies behave as those of the heat flow. More precisely, for all $\alpha \geq 0$, we have

$$(2^{2j}t)^{\alpha/2} \|Z_j(t)\|_{L^2} \leq C (2^{2j}t)^{\alpha/2} e^{-c2^{2j}t} \|Z_j(0)\|_{L^2}, \quad t \geq 0, \quad j < 0.$$

Hence, since the function $x^{\alpha/2} e^{-x}$ is bounded on \mathbb{R}_+ , we eventually get for all $s \in \mathbb{R}$,

$$(94) \quad t^{\alpha/2} \|Z(t)\|_{\dot{B}_{2,1}^{s+\alpha}}^\ell \leq C_\alpha \|Z_0\|_{\dot{B}_{2,1}^s}^\ell, \quad t \geq 0.$$

We note that, as for the free heat equation, in order to obtain some decay for the low frequencies, a shift a regularity is needed. This is the reason why it is wise to make an additional assumption (e.g. some negative regularity) on the initial data to eventually get some decay rate for the norms we considered before for the global solutions to (50). In fact, to compare our results with the classical ones in the literature, one can introduce another family of homogeneous Besov spaces, namely the sets $\dot{B}_{2,\infty}^s$ of tempered distributions z on \mathbb{R}^d satisfying

$$(95) \quad \|z\|_{\dot{B}_{2,\infty}^s} \triangleq \sup_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j z\|_{L^2} < \infty \quad \text{and} \quad \lim_{j \rightarrow -\infty} \|\chi(2^{-j}D)z\|_{L^\infty} = 0.$$

Owing to the critical embedding

$$\|z\|_{\dot{B}_{2,\infty}^{\frac{d}{2} - \frac{d}{p}}(\mathbb{R}^d)} \lesssim \|z\|_{L^p(\mathbb{R}^d)}, \quad 1 \leq p \leq 2,$$

making assumptions in spaces $\dot{B}_{2,\infty}^s$ with a negative s is *weaker* than in the pioneering works on decay estimates [28] where the initial data were assumed to be in L^1 (this corresponds to the endpoint value $\sigma_1 = d/2$) or (see [33]) in L^p for some $p \in [1, 2)$ (take $\sigma_1 = d/p - d/2$).

This motivates the following statement that we shall prove in the rest of the section:

Theorem 3.1. *Let the assumptions of Theorem 2.1 be in force, and assume in addition that $Z_0 \in \dot{B}_{2,\infty}^{-\sigma_1}$ for some σ_1 in $(1 - d/2, d/2]$. Let $\alpha_1 \triangleq (\sigma_1 + d/2 - 1)/2$ and $c_0 \triangleq (\|Z_0\|_{\dot{B}_{2,\infty}^{-\sigma_1}} + \|Z_0\|_{\tilde{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}})^{-1/\alpha_1}$.*

Then, the global solution Z constructed in Theorem 2.1 also belongs to $L^\infty(\mathbb{R}_+; \dot{B}_{2,\infty}^{-\sigma_1})$, and there exists a constant C_0 that may be computed in terms of c_0 such that

$$(1 + c_0 t)^{\alpha_1} \|Z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell + (1 + c_0 t)^{2\alpha_1} (\|Z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h + \|\partial_t Z_2(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell) \leq C_0 \|Z_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}}.$$

Remark 3.1. Under the (stronger) structure assumptions of Theorem 2.2, one can prove a similar result assuming only that σ_1 is in wider range $(-d/2, d/2]$. The inequality we eventually get is

$$(1 + c_0 t)^{\alpha_1} \|Z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + (1 + c_0 t)^{2\alpha_1} \|Z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h + (1 + c_0 t)^{2\alpha_1} \|\partial_t Z_2(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell \leq C_0 \|Z_0\|_{\tilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}}$$

with $\alpha_1 \triangleq (\sigma_1 + d/2)/2$ and $c_0 \triangleq (\|Z_0\|_{\dot{B}_{2,\infty}^{-\sigma_1}} + \|Z_0\|_{\tilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}})^{-1/\alpha_1}$.

Remark 3.2. Even though the negative Besov space assumption is weaker than in e.g. [33], the obtained decay rates are the same ones. Note also that $\|Z_0\|_{\dot{B}_{2,\infty}^{-\sigma_1}}$ can be arbitrarily large: only $\|Z_0\|_{\tilde{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}}$ has to be small.

The linear decay rate for low frequencies turns out to be the correct one for the solution of the nonlinear system (50), and better (algebraic) decay rates hold true for the high frequencies and for the damped mode. At the same time, although the high frequencies of the solution of the linearized system (12) have exponential decay, it is not the case for the nonlinear system (50) owing to the coupling between the low and high frequencies through the nonlinear terms. We do not claim optimality of the above decay rates for the high frequencies but, for sure, it is very unlikely that they are exponential even for very particular initial data.

Let us briefly explain the general strategy of the proof. The starting point is to show that the additional negative regularity is propagated for all time (with a time-independent control). Then, we shall combine it with Inequality (84) and an interpolation argument so as to exhibit a decay inequality for $\|Z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}$. The rate that we shall get in this way turns out to be precisely the one that was expected from our linear analysis in (94). Then, interpolating with the estimate in the negative space will enable us to capture optimal decay rates for intermediate norms $\|Z(t)\|_{\dot{B}_{2,1}^s}$.

To the best of our knowledge the idea of combining a Lyapunov inequality with dissipation and interpolation to get (optimal) decay rates originates from the work by J. Nash on parabolic equations in [29]⁷. Implementing it on other equations in a functional framework close to ours is rather recent. The overall strategy is well explained in a work by Y. Guo and Y. Wang [19] devoted to the Boltzmann equation and the compressible Navier-Stokes equations in the Sobolev spaces setting, and Z. Xin and J. Xu in [37] used the same method to prove decay estimates for the compressible Navier-Stokes equations in the critical regularity framework. In the context of partially dissipative systems, the idea of prescribing additional integrability in terms of negative Besov norms instead of Lebesgue ones seems to originate from a paper by J. Xu and S. Kawashima [40].

Finally, let us emphasize that it is possible to do without a Lyapunov functional (like we did in e.g. [17]) but, somehow, the proof is more technical and less ‘elegant’.

⁷Special thanks to L.-M. Rodrigues for pointing out this reference to us.

3.1. Propagation of negative regularity. In order to prove that the regularity in $\dot{B}_{2,\infty}^{-\sigma_1}$ is propagated for all time, let us start from the equation of Z_j written in the following way:

$$\partial_t Z_j + \sum_{k=1}^d A_k(Z) \partial_k Z_j + B Z_j = \sum_{k=1}^d [A_k(Z), \dot{\Delta}_j] \partial_k Z.$$

Taking the L^2 scalar product with Z_j and using (51) yields

$$(96) \quad \frac{1}{2} \frac{d}{dt} \|Z_j\|_{L^2}^2 + c \|Z_{2,j}\|_{L^2}^2 \leq \sum_{k=1}^d \|[A_k, \dot{\Delta}_j] Z\|_{L^2} \|Z_j\|_{L^2}.$$

One can show (combine the commutator inequalities of [2, Chap. 2] with (70)) that

$$\sup_{j \in \mathbb{Z}} 2^{-j\sigma_1} \|[A_k(Z), \dot{\Delta}_j] \partial_k Z\|_{L^2} \leq C \|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|Z\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \quad \text{if } -\frac{d}{2} \leq -\sigma_1 < \frac{d}{2} + 1.$$

Hence, dropping the nonnegative term in the left-hand side of (96), using Lemma A.1 and taking the supremum on j yields

$$\|Z(t)\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \leq \|Z_0\|_{\dot{B}_{2,\infty}^{-\sigma_1}} + C \int_0^t \|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|Z\|_{\dot{B}_{2,\infty}^{-\sigma_1}} d\tau, \quad t \geq 0,$$

which, after applying Gronwall lemma, leads to

$$\|Z(t)\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \leq \|Z_0\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \exp\left(C \int_0^t \|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau\right).$$

Whenever Z_0 satisfies (54), the global solution of Theorem 2.1 has (small) gradient in $L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$. Hence the above inequality guarantees that Z is uniformly bounded in $\dot{B}_{2,\infty}^{-\sigma_1}$: there exists a constant C depending only on σ_1 and such that

$$(97) \quad \sup_{t \geq 0} \|Z(t)\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \leq C \|Z_0\|_{\dot{B}_{2,\infty}^{-\sigma_1}}.$$

3.2. Decay estimates for the whole solution. The starting point is Inequality (84) that is valid for all $0 \leq t_0 \leq t$, and the fact that

$$\mathcal{L} \simeq \sum_{j \in \mathbb{Z}} 2^{j(\frac{d}{2}-1)+\max(1,2^{2j})} \|Z_j\|_{L^2} \quad \text{and} \quad \mathcal{H} = \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}.$$

Being monotonous, the function \mathcal{L} is almost everywhere differentiable on \mathbb{R}_+ and Inequality (84) thus implies that

$$(98) \quad \frac{d}{dt} \mathcal{L} + c \mathcal{H} \leq 0 \quad \text{a.e. on } \mathbb{R}_+.$$

Now, if $-\sigma_1 < d/2 - 1$, then one may use the following interpolation inequality:

$$\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell \lesssim (\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^\ell)^{1-\theta_0} (\|Z\|_{\dot{B}_{2,\infty}^{-\sigma_1}}^\ell)^{\theta_0} \quad \text{with} \quad (1-\theta_0)\left(1+\frac{d}{2}\right) - \sigma_1 \theta_0 = \frac{d}{2} - 1$$

which implies, taking advantage of (97), that

$$(99) \quad \|Z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^\ell \gtrsim (\|Z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell)^{\frac{1}{1-\theta_0}} \|Z_0\|_{\dot{B}_{2,\infty}^{-\sigma_1}}^{-\frac{\theta_0}{1-\theta_0}}.$$

To handle the high frequencies of Z , we just write that, owing to (55), we have

$$(100) \quad \|Z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \gtrsim (\|Z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h)^{\frac{1}{1-\theta_0}} \|Z_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}}^{-\frac{\theta_0}{1-\theta_0}}.$$

Putting (99) and (100) together and remembering that

$$(101) \quad \mathcal{L} \simeq \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h,$$

one may thus write that for a small enough $c > 0$, we have

$$\mathcal{H} \gtrsim c_0 \mathcal{L}^{\frac{1}{1-\theta_0}} \quad \text{with} \quad c_0 \triangleq c (\|Z_0\|_{\dot{B}_{2,\infty}^{-\sigma_1}} + \|Z_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}})^{-\frac{\theta_0}{1-\theta_0}}.$$

Reverting to (98), one eventually obtains the following differential inequality:

$$\frac{d}{dt} \mathcal{L} + c_0 \mathcal{L}^{\frac{1}{1-\theta_0}} \leq 0,$$

which readily leads to

$$(102) \quad \mathcal{L}(t) \leq (1 + c_0 t)^{1-1/\theta_0} \mathcal{L}(0).$$

Now, replacing θ_0 with its value, and using (101), one can conclude that

$$(103) \quad \|Z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}} \lesssim (1 + c_0 t)^{-\alpha_1} \|Z_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}} \quad \text{with} \quad \alpha_1 \triangleq \frac{1}{2} \left(\sigma_1 + \frac{d}{2} - 1 \right).$$

As regards the low frequencies of the solution, this decay is consistent with (94) in the case $s = -\sigma_1$ and $\alpha = \sigma_1 + d/2 - 1$.

3.3. High frequency decay. From (76) and (77), we gather

$$\frac{1}{2} \frac{d}{dt} \mathcal{L}_j + c \mathcal{L}_j \lesssim 2^{-j} \|F_j\|_{L^2} \|Z_j\|_{L^2} + c_j 2^{-j(\frac{d}{2}+1)} \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^2 \sqrt{\mathcal{L}_j} \quad \text{with} \quad \sum_{j \geq 0} c_j = 1.$$

Hence, bounding F_j according to (75) yields

$$\frac{1}{2} \frac{d}{dt} \mathcal{L}_j + c \mathcal{L}_j \lesssim c_j 2^{-j(\frac{d}{2}+1)} \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} (\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}) \sqrt{\mathcal{L}_j} \quad \text{with} \quad \sum_{j \geq 0} c_j = 1,$$

whence

$$\frac{1}{2} \frac{d}{dt} (e^{ct} \sqrt{\mathcal{L}_j})^2 \lesssim c_j 2^{-j(\frac{d}{2}+1)} \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}} (e^{ct} \sqrt{\mathcal{L}_j}).$$

By time integration (viz. we use Lemma A.1), we deduce that

$$\sqrt{\mathcal{L}_j(t)} \leq e^{-ct} \sqrt{\mathcal{L}_j(0)} + C 2^{-j(\frac{d}{2}+1)} \int_0^t e^{-c(t-\tau)} c_j(\tau) \|Z(\tau)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|Z(\tau)\|_{\dot{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}} d\tau.$$

Hence, multiplying both sides by $2^{j(\frac{d}{2}+1)}$, then summing up on $j \geq 0$ and using the equivalence of the high frequency part of (79) with the norm in $\dot{B}_{2,1}^{\frac{d}{2}+1}$, we end up with

$$\|Z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \lesssim e^{-ct} \|Z_0\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h + \int_0^t e^{-c(t-\tau)} \|Z(\tau)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|Z(\tau)\|_{\dot{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}} d\tau.$$

Consequently, for all $t \geq 0$,

$$(1 + c_0 t)^{2\alpha_1} \|Z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \lesssim (1 + c_0 t)^{2\alpha_1} e^{-ct} \|Z_0\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \\ + \int_0^t \left(\frac{1 + c_0 t}{1 + c_0 \tau} \right)^{2\alpha_1} e^{-c(t-\tau)} \left((1 + c_0 \tau)^{\alpha_1} \|Z(\tau)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right)^2 d\tau.$$

Inequality (102) ensures that

$$\sup_{\tau \geq 0} (1 + c_0 \tau)^{\alpha_1} \|Z(\tau)\|_{\dot{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}} \leq C \|Z_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}}.$$

Furthermore, one can find a constant C_0 depending only on α_1 and c_0 such that

$$\int_0^t \left(\frac{1 + c_0 t}{1 + c_0 \tau} \right)^{2\alpha_1} e^{-c(t-\tau)} d\tau \leq C_0, \quad t \geq 0.$$

Hence, in the end, we get

$$(104) \quad \sup_{t \geq 0} (1 + c_0 t)^{2\alpha_1} \|Z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h \leq C_0 \|Z_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}}.$$

3.4. The decay of the damped mode. According to (86) and to (57), the damped mode $W \triangleq -L_2^{-1} \partial_t Z_2$ satisfies a relation of the form

$$\partial_t W + L_2 W \simeq Z \cdot \nabla^2 Z + \nabla Z \cdot \nabla Z + (1 + Z) \nabla W$$

and, according to (85), we have

$$W \simeq Z_2 + \nabla Z + Z \nabla Z.$$

Therefore, applying $\dot{\Delta}_j$ to the above equation, taking the L^2 scalar product with $W_j \triangleq \dot{\Delta}_j W$ and using Bernstein inequality in order to bound the last term, we get for all $j \in \mathbb{Z}$,

$$\frac{1}{2} \frac{d}{dt} \|W_j\|_{L^2}^2 + c \|W_j\|_{L^2}^2 \leq C (\|\dot{\Delta}_j((1 + Z)Z \cdot \nabla^2 Z)\|_{L^2} + \|\dot{\Delta}_j((1 + Z)\nabla Z \cdot \nabla Z)\|_{L^2} \\ + \|\dot{\Delta}_j(\nabla Z_2 \cdot Z)\|_{L^2}) \|W_j\|_{L^2} + C 2^j \|W_j\|_{L^2}^2.$$

Let us choose $j_0 \in \mathbb{Z}$ such that $C 2^{j_0} \leq c/2$ (so that the last term may be absorbed by the left-hand side). Then, using Lemma A.1, multiplying both sides by $2^{j(\frac{d}{2}-1)}$, then summing up on $j \leq j_0$, we end up with⁸

$$(105) \quad \|W(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell \leq e^{-ct} \|W_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell \\ + C \int_0^t e^{-c(t-\tau)} (\|(1 + Z)Z \otimes \nabla^2 Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell + \|(1 + Z)\nabla Z \otimes \nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell + \|Z \otimes \nabla Z_2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell) d\tau.$$

Since $d \geq 2$, the product laws (69) and (74) guarantee that

$$\|(1 + Z)Z \otimes \nabla^2 Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell + \|(1 + Z)\nabla Z \otimes \nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell \lesssim (1 + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}},$$

⁸Rigorously speaking the low frequencies that are here considered are lower than with our previous definition since it may happen that $j_0 \leq 0$. However, one may check that the high frequency decay estimate in (104) still holds if we put the threshold at some $j_0 \leq 0$: the argument we used works if summing up on $j \geq j_0$ provided we change the ‘constants’ accordingly.

which, combined with (103) and the fact that $\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}$ is small implies that

$$\|(1+Z)Z \otimes \nabla^2 Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|(1+Z)\nabla Z \otimes \nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \lesssim (1+c_0 t)^{-2\alpha_1} \mathcal{L}^2(0).$$

Similarly, we have

$$\|(1+Z)Z \otimes \nabla Z_2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \lesssim \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \lesssim \|Z\|_{\tilde{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}}^2 \lesssim (1+c_0 t)^{-2\alpha_1} \mathcal{L}^2(0).$$

Hence, using (105) and arguing as in the previous paragraph, we end up with

$$(106) \quad \sup_{t \geq 0} (1+c_0 t)^{2\alpha_1} \|W(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^\ell \leq C_0 \|Z_0\|_{\tilde{B}_{2,1}^{\frac{d}{2}-1, \frac{d}{2}+1}}.$$

In other words, the decay rate for the low frequencies of the damped mode in norm $\dot{B}_{2,1}^{\frac{d}{2}-1}$ is the same as that of the high frequencies of the whole solution.

Summing up the results of the previous paragraphs completes the proof of Theorem 3.1.

4. ON THE STRONG RELAXATION LIMIT

This section is devoted to the study of a singular limit problem for the following class of partially dissipative hyperbolic systems:

$$(107) \quad \partial_t Z^\varepsilon + \sum_{k=1}^d A^k(Z^\varepsilon) \partial_k Z^\varepsilon + \frac{BZ^\varepsilon}{\varepsilon} = 0,$$

where, denoting $\bar{A}_{\ell m}^k \triangleq A_{\ell m}^k(0)$ and $\tilde{A}_{\ell m}^k(Z) \triangleq A_{\ell m}^k(Z) - \bar{A}_{\ell m}^k$, we assume that for all $k \in \{1, \dots, d\}$:

- (1) $\bar{A}_{11}^k = 0$, and \tilde{A}_{11}^k is linear with respect to Z_2 and independent of Z_1 ,
- (2) \tilde{A}_{12}^k and \tilde{A}_{21}^k are linear with respect to Z_1 and independent of Z_2 ,
- (3) \tilde{A}_{22}^k is linear with respect to Z ,
- (4) Condition (SK) is satisfied by the pair $(A(\xi), B)$ with $A(\xi)$ defined in (13), at every point $\xi \in \mathbb{R}^d$.

The linearity assumption is here just for simplicity as well as the fact that there is no 0-order nonlinear term. At the same time, assuming that A_{12}^k and A_{21}^k (resp. A_{11}^k) only depend on Z_1 (resp. Z_2) is very helpful, if not essential. We shall see that it is satisfied by the compressible Euler equations written in terms of the sound speed (see (5)).

We want to study the so-called ‘strong relaxation limit’, that is whether the global solutions of (107) constructed before tend to satisfy some limit system when ε goes to 0.

A hasty analysis suggests that the part of the solution that experiences direct dissipation, namely Z_2^ε with the notation of the previous sections, tends to 0 with a characteristic time of order ε and that, consequently, Z_1^ε tends to be time independent (since, for all $k \in \{1, \dots, d\}$, we have $\bar{A}_{11}^k = 0$ and A_{11}^k is independent of Z_1). To some extent this will prove to be true but, even for the simple case of the linearized one-dimensional compressible Euler equations, the situation is more complex than expected. Indeed, consider

$$(108) \quad \begin{cases} \partial_t a + \partial_x u = 0, \\ \partial_t u + \partial_x a + \varepsilon^{-1} u = 0. \end{cases}$$

In the Fourier space, this system translates into

$$\frac{d}{dt} \begin{pmatrix} \widehat{a} \\ \widehat{u} \end{pmatrix} + \begin{pmatrix} 0 & i\xi \\ i\xi & \varepsilon^{-1} \end{pmatrix} \begin{pmatrix} \widehat{a} \\ \widehat{u} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- In low frequencies $|\xi| < (2\varepsilon)^{-1}$, the matrix $A(\xi)$ of this system has the following two real eigenvalues:

$$\lambda^\pm(\xi) = \frac{1}{2\varepsilon} \left(1 \pm \sqrt{1 - (2\varepsilon\xi)^2} \right).$$

For ξ going to 0, we observe that

$$\lambda^+(\xi) \sim \varepsilon^{-1} \quad \text{and} \quad \lambda^-(\xi) \sim \varepsilon\xi^2.$$

This means that one of the modes of the system is indeed damped with coefficient ε^{-1} but that the overall behavior of solutions of the system is like for the inviscid limit (or for the heat equation with vanishing diffusion).

- In high frequencies $|\xi| > (2\varepsilon)^{-1}$, the matrix $A(\xi)$ has the following two complex conjugated eigenvalues:

$$\lambda^\pm(\xi) = \frac{1}{2\varepsilon} \left(1 \pm i\sqrt{(2\varepsilon\xi)^2 - 1} \right).$$

Clearly, $\operatorname{Re} \lambda^\pm(\xi) = (2\varepsilon)^{-1}$ and $\operatorname{Im} \lambda^\pm(\xi) \sim i\xi$ for $\xi \rightarrow \infty$. Hence, there is indeed dissipation with characteristic time ε for the high frequencies of the solution.

The ‘low frequency regime’ is expected to dominate when $\varepsilon \rightarrow 0$, as it corresponds to $|\xi| \lesssim \varepsilon^{-1}$. Consequently, the overall behavior of System (108) might be similar to that of the heat flow with diffusion ε , and one can wonder if the high relaxation limit is analogous to the inviscid limit⁹. However, we have to keep in mind that the low frequencies of the ‘damped mode’ (that here corresponds to the combination $u + \varepsilon\partial_x a$) undergo a much stronger dissipation. This is of course an element that one has to take into consideration.

Based on this simple example, it looks that in order to investigate the high relaxation limit, it is suitable to use a functional framework that non only reflects the different behavior of the low and high frequencies (with threshold being located around ε^{-1}) but also emphasizes the better properties of the damped mode.

4.1. A ‘cheap’ result of convergence. Let us revert to the general class of Systems (107) supplemented with initial data Z_0^ε . The structure assumptions that we made at the beginning of the section enable us to apply Theorem 2.2. In this Subsection, we shall take advantage of it and of elementary scaling considerations so as to establish that both $Z_1^\varepsilon - Z_{1,0}^\varepsilon$ and Z_2^ε converge strongly to 0 for suitable norms. The reader may refer to the next subsection for a more accurate result.

The starting observation is the following change of time and space scale:

$$(109) \quad \widetilde{Z}(t, x) \triangleq Z^\varepsilon(\varepsilon t, \varepsilon x).$$

Clearly, Z^ε satisfies (107) if and only if \widetilde{Z} satisfies (57).

⁹This phenomenon that is well known in physics is sometimes called *overdamping*.

The following property of homogeneous Besov norms is well known (see [2, Chap. 2]):

$$(110) \quad \|z(\varepsilon \cdot)\|_{\dot{B}_{2,1}^s} \simeq \varepsilon^{s-d/2} \|z\|_{\dot{B}_{2,1}^s}.$$

By adapting the proof therein, one can prove that

$$(111) \quad \|z(\varepsilon \cdot)\|_{\dot{B}_{2,1}^s}^\ell \simeq \varepsilon^{s-d/2} \|z\|_{\dot{B}_{2,1}^s}^{\ell, \varepsilon^{-1}} \quad \text{and} \quad \|z(\varepsilon \cdot)\|_{\dot{B}_{2,1}^s}^h \simeq \varepsilon^{s-d/2} \|z\|_{\dot{B}_{2,1}^s}^{h, \varepsilon^{-1}}$$

where we have used the notation

$$(112) \quad \|z\|_{\dot{B}_{2,1}^s}^{\ell, \alpha} \triangleq \sum_{j \in \mathbb{Z}, 2^j < \alpha} 2^{js} \|\Delta_j z\|_{L^2} \quad \text{and} \quad \|z\|_{\dot{B}_{2,1}^s}^{h, \alpha} \triangleq \sum_{j \in \mathbb{Z}, 2^j \geq \alpha} 2^{js} \|\Delta_j z\|_{L^2}.$$

Putting together (110), (111), the change of unknowns (109) and Theorem 2.2 readily gives the following global existence result that is valid for all $\varepsilon > 0$.

Theorem 4.1. *There exists a positive constant α such that for all $\varepsilon > 0$ and data Z_0^ε satisfying*

$$(113) \quad \mathcal{Z}_0^\varepsilon \triangleq \|Z_0^\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{\ell, \varepsilon^{-1}} + \varepsilon \|Z_0^\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h, \varepsilon^{-1}} \leq \alpha,$$

System (107) supplemented with initial data Z_0^ε admits a unique global-in-time solution Z^ε satisfying the inequality

$$(114) \quad \mathcal{Z}^\varepsilon(t) \leq C \mathcal{Z}_0^\varepsilon \quad \text{with}$$

$$\begin{aligned} \mathcal{Z}^\varepsilon(t) \triangleq & \|Z^\varepsilon\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^{\ell, \varepsilon^{-1}} + \varepsilon \|Z^\varepsilon\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}+1})}^{h, \varepsilon^{-1}} + \varepsilon \|Z_1^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+2})}^{\ell, \varepsilon^{-1}} + \|Z_2^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^{\ell, \varepsilon^{-1}} + \varepsilon^{-1/2} \|Z_2^\varepsilon\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^{\ell, \varepsilon^{-1}} \\ & + \|Z^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|\partial_t Z_2^\varepsilon\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^\ell. \end{aligned}$$

The above theorem implies that $Z_1^\varepsilon \rightarrow Z_1^\varepsilon(0)$ and that $Z_2^\varepsilon \rightarrow 0$ when $\varepsilon \rightarrow 0$. Indeed, from the definition (112), it is obvious that for all $\eta > 0$, $\beta \geq 0$ and $s \in \mathbb{R}$, we have

$$(115) \quad \|z\|_{\dot{B}_{2,1}^{s+\beta}}^{\ell, \eta} \lesssim \eta^\beta \|z\|_{\dot{B}_{2,1}^s}^{\ell, \eta} \quad \text{and} \quad \|z\|_{\dot{B}_{2,1}^{s-\beta}}^{h, \eta} \lesssim \eta^{-\beta} \|z\|_{\dot{B}_{2,1}^s}^{h, \eta}.$$

Hence, using (114) and Hölder inequality yields

$$(116) \quad \|Z^\varepsilon\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})}^{h, \varepsilon^{-1}} \leq C \varepsilon^{1/2} \mathcal{Z}_0^\varepsilon.$$

Thanks to (113) and, again, to (114), this allows to get

$$(117) \quad \|Z_2^\varepsilon\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \leq C \alpha \varepsilon^{1/2}.$$

In order to justify that $Z_1^\varepsilon \rightarrow Z_1^\varepsilon(0)$, one may bound $\partial_t Z_1^\varepsilon$ through (107) remembering that the blocks A_{11}^k are linear with respect to Z_2^ε . From the product law (69), and from (114) and (117), we get

$$\|\partial_t Z_1^\varepsilon\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})} \lesssim \|Z_2^\varepsilon\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \|Z^\varepsilon\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \leq C \alpha^2 \varepsilon^{1/2},$$

and thus

$$(118) \quad \|Z_1^\varepsilon(t) - Z_{1,0}^\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \leq C \alpha^2 (\varepsilon t)^{1/2} \quad \text{for all } t \geq 0.$$

In conclusion, Z_2^ε tends to 0 in $L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$ with rate of convergence $\varepsilon^{1/2}$, and $Z_1^\varepsilon - Z_{1,0}^\varepsilon$ converges to 0 in $L^\infty([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1})$ with rate $(\varepsilon T)^{1/2}$, for all $T > 0$.

4.2. Connections with porous media-like equations. In order to exhibit richer dynamics in the asymptotics $\varepsilon \rightarrow 0$, one may perform the following ‘diffusive’ rescaling:

$$(119) \quad (\tilde{Z}_1^\varepsilon, \tilde{Z}_2^\varepsilon)(\tau, x) = (Z_1^\varepsilon, \varepsilon^{-1} Z_2^\varepsilon)(\varepsilon^{-1} \tau, x).$$

Dropping the exponents ε for better readability, we get the following system for $(\tilde{Z}_1, \tilde{Z}_2)$:

$$(120) \quad \begin{cases} \partial_\tau \tilde{Z}_1 + \sum_{k=1}^d \tilde{A}_{11}^k(\tilde{Z}_2) \partial_k \tilde{Z}_1 + \sum_{k=1}^d (\bar{A}_{12}^k + \tilde{A}_{12}^k(\tilde{Z}_1)) \partial_k \tilde{Z}_2 = 0, \\ \varepsilon^2 \partial_\tau \tilde{Z}_2 + \varepsilon \sum_{k=1}^d (\bar{A}_{22}^k + \tilde{A}_{22}^k(\tilde{Z}_1, \varepsilon \tilde{Z}_2)) \partial_k \tilde{Z}_2 + \sum_{k=1}^d (\bar{A}_{21}^k + \tilde{A}_{21}^k(\tilde{Z}_1)) \partial_k \tilde{Z}_1 + L_2 \tilde{Z}_2 = 0. \end{cases}$$

From the second line, one can expect

$$(121) \quad \tilde{W} \triangleq \tilde{Z}_2 + L_2^{-1} \left(\sum_{k=1}^d (\bar{A}_{21}^k + \tilde{A}_{21}^k(\tilde{Z}_1)) \partial_k \tilde{Z}_1 \right) \longrightarrow 0.$$

In order to find out what could be the limit system for \tilde{Z}_1 , let us systematically express \tilde{Z}_2 in terms of \tilde{W} and \tilde{Z}_1 by means of (121) in the first line of (120). We get

$$\begin{aligned} \partial_\tau \tilde{Z}_1 + \sum_k (\bar{A}_{12}^k + \tilde{A}_{12}^k(\tilde{Z}_1)) \partial_k \tilde{W} + \tilde{A}_{11}^k(\tilde{W}) \partial_k \tilde{Z}_1 \\ + \sum_{k,\ell} (\bar{A}_{12}^k + \tilde{A}_{12}^k(\tilde{Z}_1)) L_2^{-1} \partial_k ((\bar{A}_{21}^\ell + \tilde{A}_{21}^\ell(\tilde{Z}_1)) \partial_\ell \tilde{Z}_1) \\ + \sum_{k,\ell} \tilde{A}_{11}^k (L_2^{-1} (\bar{A}_{21}^\ell + \tilde{A}_{21}^\ell(\tilde{Z}_1)) \partial_\ell \tilde{Z}_1) \partial_k \tilde{Z}_1 = 0. \end{aligned}$$

Introducing the following second order operator:

$$(122) \quad \mathcal{A} \triangleq \sum_{k,\ell} \bar{A}_{12}^k L_2^{-1} \bar{A}_{21}^\ell \partial_k \partial_\ell,$$

the above equation may be rewritten:

$$(123) \quad \partial_\tau \tilde{Z}_1 + \mathcal{A} \tilde{Z}_1 + Q_1(\tilde{Z}_1, \nabla^2 \tilde{Z}_1) + Q_2(\nabla \tilde{Z}_1, \nabla \tilde{Z}_1) \\ + T_1(\tilde{Z}_1, \nabla \tilde{Z}_1, \nabla \tilde{Z}_1) + T_2(\tilde{Z}_1, \tilde{Z}_1, \nabla^2 \tilde{Z}_1) = S$$

where, Q_1 , Q_2 (resp. T_1 , T_2) are bilinear (resp. trilinear) expressions that may be computed in terms of the coefficients of the matrices \tilde{A}_{11}^k , \tilde{A}_{12}^k and of L_2 , and

$$(124) \quad S \triangleq - \sum_{k=1}^d (\bar{A}_{12}^k + \tilde{A}_{12}^k(\tilde{Z}_1)) \partial_k \tilde{W} - \sum_{k=1}^d \tilde{A}_{11}^k(\tilde{W}) \partial_k \tilde{Z}_1.$$

Consequently, if (121) is true, then we expect \tilde{Z}_1 to tend to \tilde{N} with \tilde{N} satisfying

$$(125) \quad \partial_\tau \tilde{N} + \mathcal{A} \tilde{N} + Q_1(\tilde{N}, \nabla^2 \tilde{N}) + Q_2(\nabla \tilde{N}, \nabla \tilde{N}) + T_1(\tilde{N}, \nabla \tilde{N}, \nabla \tilde{N}) + T_2(\tilde{N}, \tilde{N}, \nabla^2 \tilde{N}) = 0.$$

Note that, as a consequence of Lemma A.3 in Appendix, and since we assumed both Condition (SK) and that $\bar{A}_{11}^k = 0$ for all $k \in \{1, \dots, d\}$, (125) is a quasilinear (scalar) parabolic equation.

Before justifying the above heuristics in the general case, let us again consider the compressible Euler equations, that is

$$(126) \quad \begin{cases} \partial_t \varrho^\varepsilon + \operatorname{div}(\varrho^\varepsilon v^\varepsilon) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t(\varrho^\varepsilon v^\varepsilon) + \operatorname{div}(\varrho^\varepsilon v^\varepsilon \otimes v^\varepsilon) + \nabla(P(\varrho^\varepsilon)) + \varepsilon^{-1} \varrho^\varepsilon v^\varepsilon = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \end{cases}$$

Under the isentropic assumption

$$(127) \quad P(z) = Az^\gamma \quad \text{with } \gamma > 1 \quad \text{and } A > 0,$$

the above system enters in the class (107) if reformulated in terms of $(c^\varepsilon, \varrho^\varepsilon)$, where

$$(128) \quad c^\varepsilon \triangleq \frac{(\gamma A)^{\frac{1}{2}}}{\tilde{\gamma}} (\varrho^\varepsilon)^{\tilde{\gamma}} \quad \text{and} \quad \tilde{\gamma} \triangleq \frac{\gamma - 1}{2}.$$

Indeed, we get:

$$(129) \quad \begin{cases} \partial_t c^\varepsilon + v^\varepsilon \cdot \nabla c^\varepsilon + \tilde{\gamma} c^\varepsilon \operatorname{div} v^\varepsilon = 0, \\ \partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon + \tilde{\gamma} c^\varepsilon \nabla c^\varepsilon + \varepsilon^{-1} v^\varepsilon = 0. \end{cases}$$

So, if we set $\bar{c} \triangleq \frac{(\gamma A)^{\frac{1}{2}}}{\tilde{\gamma}} (\bar{\varrho})^{\tilde{\gamma}}$, then Conditions (1) to (4) below (107) are satisfied with $Z_1^\varepsilon = c^\varepsilon - \bar{c}$ and $Z_2^\varepsilon = v^\varepsilon$.

Now, performing the diffusive rescaling:

$$(130) \quad (\varrho^\varepsilon, v^\varepsilon)(t, x) = (\tilde{\varrho}^\varepsilon, \varepsilon \tilde{v}^\varepsilon)(\varepsilon t, x),$$

System (126) becomes

$$(131) \quad \begin{cases} \partial_\tau \tilde{\varrho}^\varepsilon + \operatorname{div}(\tilde{\varrho}^\varepsilon \tilde{v}^\varepsilon) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \varepsilon^2 \partial_\tau(\tilde{\varrho}^\varepsilon \tilde{v}^\varepsilon) + \varepsilon \operatorname{div}(\tilde{\varrho}^\varepsilon \tilde{v}^\varepsilon \otimes \tilde{v}^\varepsilon) + \nabla(P(\tilde{\varrho}^\varepsilon)) + \tilde{\varrho}^\varepsilon \tilde{v}^\varepsilon = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \end{cases}$$

In light of the second equation, it is expected that

$$\nabla(P(\tilde{\varrho}^\varepsilon)) + \tilde{\varrho}^\varepsilon \tilde{v}^\varepsilon \rightarrow 0 \quad \text{when } \varepsilon \rightarrow 0,$$

and thus that $\tilde{\varrho}^\varepsilon$ converges to some solution \tilde{N} of the porous media equation:

$$(132) \quad \partial_\tau \tilde{N} - \Delta(P(\tilde{N})) = 0.$$

The general result we shall prove for Systems (107) reads as follows for the particular case of the isentropic Euler equations¹⁰:

Theorem 4.2. *Consider the Euler equations with relaxation (126) in \mathbb{R}^d (with $d \geq 1$) with pressure law (127) and initial data $(\varrho_0^\varepsilon, v_0^\varepsilon)$ such that $(\varrho^\varepsilon - \bar{\varrho}) \in \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}$ and $v_0^\varepsilon \in \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}$. There exists $\alpha > 0$ independent of ε such that if*

$$(133) \quad \|(\varrho_0^\varepsilon - \bar{\varrho}, v_0^\varepsilon)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{\ell, \varepsilon^{-1}} + \varepsilon \|(\varrho_0^\varepsilon - \bar{\varrho}, v_0^\varepsilon)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h, \varepsilon^{-1}} \leq \alpha$$

¹⁰A statement in the same spirit, but allowing for Besov spaces constructed on L^p may be found in [12].

then (126) supplemented with $(\varrho_0^\varepsilon, v_0^\varepsilon)$ has a unique solution $(\varrho^\varepsilon, v^\varepsilon)$ with $(\varrho^\varepsilon - \bar{\varrho}, v^\varepsilon) \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1})$ satisfying in addition

$$(134) \quad \begin{aligned} & \|(\varrho^\varepsilon - \bar{\varrho}, v^\varepsilon)\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})}^{\ell, \varepsilon^{-1}} + \varepsilon \|(\varrho^\varepsilon - \bar{\varrho}, v^\varepsilon)\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})}^{h, \varepsilon^{-1}} + \varepsilon^{1/2} \|\varrho^\varepsilon - \bar{\varrho}\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})} \\ & + \|v^\varepsilon\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})} + \varepsilon^{-1/2} \|v^\varepsilon\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})}^{\ell, \varepsilon^{-1}} + \|\varepsilon^{-1} v^\varepsilon + (\varrho^\varepsilon)^{-1} \nabla(P(\varrho^\varepsilon))\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})}^{\ell, \varepsilon^{-1}} \leq C\alpha. \end{aligned}$$

Furthermore, for any \tilde{N}_0 in $\dot{B}_{2,1}^{\frac{d}{2}}$ such that $\|\tilde{N}_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq \alpha$, Equation (132) has a unique solution \tilde{N} in the space $\mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}) \cap L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+2})$ satisfying for all $t \geq 0$,

$$\|\tilde{N}(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \int_0^t \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} d\tau \leq C \|\tilde{N}_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

Finally, if one denotes by $(\tilde{\varrho}^\varepsilon, \tilde{v}^\varepsilon)$ the rescaled solution of the Euler equations defined through (130) and assumes in addition that

$$\|\tilde{N}_0 - \tilde{\varrho}_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \leq C\varepsilon,$$

then we have

$$(135) \quad \left\| \tilde{v}^\varepsilon + \frac{\nabla(P(\tilde{\varrho}^\varepsilon))}{\tilde{\varrho}^\varepsilon} \right\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} + \|\tilde{N} - \tilde{\varrho}^\varepsilon\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\tilde{N} - \tilde{\varrho}^\varepsilon\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})} \leq C\varepsilon.$$

Proof. Let us assume for a while that $\varepsilon = 1$ so that one can readily take advantage of Theorem 2.3. As a first, we want to translate Theorem 2.3 in terms of ϱ , where c and ϱ (resp. \bar{c} and $\bar{\varrho}$) are interrelated through (128).

On the one hand, Inequality (61), the property of interpolation in Besov spaces and Hölder inequality with respect to the time variable imply that

$$\|c - \bar{c}\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})} \leq C \|(c_0 - \bar{c}, v_0)\|_{\tilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}}.$$

On the other hand, using the fact that the composition inequality (70) is actually valid for all positive Besov exponents (see e.g. [2][Chap. 2]), we may write that

$$\|c - \bar{c}\|_{\dot{B}_{2,1}^{\frac{d}{2}+\alpha}} \approx \|\varrho - \bar{\varrho}\|_{\dot{B}_{2,1}^{\frac{d}{2}+\alpha}} \quad \text{for } \alpha = 0, 1.$$

Finally, we note that $\partial_t v = -v - \varrho^{-1} \nabla(P(\varrho)) - v \cdot \nabla v$ and that

$$\|v \cdot \nabla v\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \leq C \|v\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla v\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})}.$$

Therefore, the last term of $\partial_t v$ may be ‘omitted’ in Inequality (61), and we get

$$(136) \quad \begin{aligned} & \|(\varrho - \bar{\varrho}, v)\|_{L^\infty(\mathbb{R}_+; \tilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1})} + \|\varrho - \bar{\varrho}\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})}^\ell + \|v\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})} + \|v\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})}^\ell \\ & + \|v + \varrho^{-1} \nabla(P(\varrho))\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \leq C \|(c_0 - \bar{c}, v_0)\|_{\tilde{B}_{2,1}^{\frac{d}{2}, \frac{d}{2}+1}}. \end{aligned}$$

Now, for general $\varepsilon > 0$, performing the rescaling (109) and remembering the equivalences (110) and (111) gives the first part of Theorem 4.2.

After performing the diffusive rescaling (130), the rescaled pair $(\tilde{\varrho}^\varepsilon, \tilde{v}^\varepsilon)$ satisfies

$$\partial_t \tilde{\varrho}^\varepsilon - \Delta(P(\tilde{\varrho}^\varepsilon)) = -\operatorname{div}(\tilde{\varrho}^\varepsilon \tilde{W}^\varepsilon) \quad \text{with} \quad \tilde{W}^\varepsilon \triangleq \tilde{v}^\varepsilon + \frac{\nabla(P(\tilde{\varrho}^\varepsilon))}{\tilde{\varrho}^\varepsilon}.$$

Thanks to (110), the bound for the last term in (136) translates into

$$(137) \quad \|\tilde{W}^\varepsilon\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \leq C\alpha\varepsilon,$$

which completes the proof of (134).

Proving that $\tilde{\varrho}^\varepsilon$ tends to some solution \tilde{N} of (132) may be done exactly as in the general case presented below in Theorem 4.3. \square

Let us finally turn to the study of the strong relaxation limit in the general case. The main result we shall get reads as follows:

Theorem 4.3. *Assume that¹¹ $d \geq 2$ and consider a system of type (120) for some $\varepsilon > 0$. Let the structure hypotheses listed below (107) be in force. There exists a positive constant α (independent of ε) such that for any initial data $\tilde{N}_0 \in \dot{B}_{2,1}^{\frac{d}{2}}$ for (125) and $\tilde{Z}_0^\varepsilon \in \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}$ for (120) satisfying*

$$(138) \quad \|\tilde{N}_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq \alpha,$$

$$(139) \quad \mathcal{Z}_0^\varepsilon \triangleq \|\tilde{Z}_{0,1}^\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{\ell, \varepsilon^{-1}} + \varepsilon \|\tilde{Z}_{0,2}^\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{\ell, \varepsilon^{-1}} + \varepsilon \|\tilde{Z}_{0,1}^\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h, \varepsilon^{-1}} + \varepsilon^2 \|\tilde{Z}_{0,2}^\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{h, \varepsilon^{-1}} \leq \alpha,$$

System (125) admits a unique solution \tilde{N} in the space

$$\mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}) \cap L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+2}),$$

satisfying for all $t \geq 0$,

$$(140) \quad \|\tilde{N}(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \int_0^t \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} d\tau \leq C \|\tilde{N}_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}},$$

and System (120) has a unique global-in-time solution \tilde{Z}^ε in $\mathcal{C}(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1})$ such that

$$(141) \quad \begin{aligned} & \|\tilde{Z}_1^\varepsilon\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})}^{\ell, \varepsilon^{-1}} + \varepsilon \|\tilde{Z}_2^\varepsilon\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})}^{\ell, \varepsilon^{-1}} + \varepsilon \|\tilde{Z}_1^\varepsilon\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})}^{h, \varepsilon^{-1}} \\ & + \varepsilon^2 \|\tilde{Z}_2^\varepsilon\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})}^{h, \varepsilon^{-1}} + \|\tilde{Z}_1^\varepsilon\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+2})}^{\ell, \varepsilon^{-1}} + \varepsilon^{-1} \|\tilde{Z}_1^\varepsilon\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})}^{h, \varepsilon^{-1}} + \|\tilde{Z}_2^\varepsilon\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})} \\ & + \|\tilde{Z}_2^\varepsilon\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})}^{\ell, \varepsilon^{-1}} + \varepsilon^{-1} \|\tilde{W}^\varepsilon\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \leq C \mathcal{Z}_0^\varepsilon, \end{aligned}$$

where \tilde{W}^ε has been defined in (121).

¹¹The one-dimensional case is tractable either under specific assumptions on the nonlinearities that are satisfied by the Euler equations, or in a slightly different functional framework. More details may be found in [12].

If, in addition,

$$\|\tilde{N}_0 - \tilde{Z}_{1,0}^\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \leq C\varepsilon,$$

then we have

$$(142) \quad \|\tilde{N} - \tilde{Z}_1^\varepsilon\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\tilde{N} - \tilde{Z}_1^\varepsilon\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})} \leq C\varepsilon.$$

Proof. That (120) supplemented with initial data \tilde{Z}_0 admits a unique global solution satisfying (141) follows from Theorem 2.2 after suitable rescaling. Indeed, if we make the change of unknowns:

$$(143) \quad (\tilde{Z}_1, \tilde{Z}_2)(\tau, x) = (\check{Z}_1, \frac{\check{Z}_2}{\varepsilon}) \left(\frac{\tau}{\varepsilon^2}, \frac{x}{\varepsilon} \right),$$

then we discover that \tilde{Z} satisfies (120) if and only if \check{Z} is a solution to (57). Then, taking advantage of the equivalence of norms pointed out in (110) and (111) gives the desired global existence result and (141) up to the last term since defining the damped mode as in (85) would lead to the function

$$(144) \quad \tilde{Z}_2 + L_2^{-1} \sum_{k=1}^d (\bar{A}_{21}^k + \tilde{A}_{21}^k(\tilde{Z}_1)) \partial_k \tilde{Z}_1 + \varepsilon L_2^{-1} \sum_{k=1}^d (\bar{A}_{22}^k + \tilde{A}_{22}^k(\tilde{Z}_1, \varepsilon \tilde{Z}_2)) \partial_k \tilde{Z}_2.$$

However, combining Inequality (141) (without the last term of course) with (115) ensures that

$$\|\tilde{Z}_1\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} + \varepsilon \|\tilde{Z}_2\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} + \|\nabla \tilde{Z}_2\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \lesssim \mathcal{Z}_0^\varepsilon.$$

Hence the last term of (144) is of order ε in $L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$, and \tilde{W} does satisfy (141).

In order to prove the convergence of \tilde{Z}_1 to \tilde{N} , let us first verify that S defined in (124) is of order ε in $L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})$. As $d \geq 2$, it is just a matter of taking advantage of the product law (69) to get

$$\|S\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \lesssim (1 + \|\tilde{Z}_1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|\nabla \tilde{W}\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|\tilde{W}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla \tilde{Z}_1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}.$$

Hence,

$$\|S\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})} \lesssim (1 + \|\tilde{Z}_1\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})}) \|\tilde{W}\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})}$$

and using (141) and the smallness of the initial data thus yields

$$(145) \quad \|S\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})} \leq C\varepsilon\alpha.$$

Let us next briefly justify that any data \tilde{N}_0 satisfying (138) gives rise to a unique global solution \tilde{N} of (125) in $\mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}) \cap L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+2})$ satisfying (140). In fact, since the operator \mathcal{A} is strongly elliptic, the parabolic estimates in Besov spaces with last index 1 recalled in Proposition A.1 ensure that any smooth enough global solution \tilde{N} satisfies for all $t \geq 0$,

$$\begin{aligned} \|\tilde{N}(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \int_0^t \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} d\tau &\lesssim \|\tilde{N}_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \int_0^t \left(\|Q_1(\tilde{N}, \nabla^2 \tilde{N})\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right. \\ &\quad \left. + \|Q_2(\nabla \tilde{N}, \nabla \tilde{N})\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|T_1(\tilde{N}, \nabla \tilde{N}, \nabla \tilde{N})\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|T_2(\tilde{N}, \tilde{N}, \nabla^2 \tilde{N})\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) d\tau. \end{aligned}$$

Using the stability of the space $\dot{B}_{2,1}^{\frac{d}{2}}$ by product and an obvious interpolation inequality, the nonlinear terms may be estimated as follows:

$$\begin{aligned} \|Q_1(\tilde{N}, \nabla^2 \tilde{N})\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla^2 \tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}, \\ \|Q_2(\nabla \tilde{N}, \nabla \tilde{N})\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|\nabla \tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \lesssim \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}, \\ \|T_1(\tilde{N}, \nabla \tilde{N}, \nabla \tilde{N})\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla \tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \lesssim \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}, \\ \|T_2(\tilde{N}, \tilde{N}, \nabla^2 \tilde{N})\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \|\nabla^2 \tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \lesssim \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}. \end{aligned}$$

Hence, we have for all $t \geq 0$,

$$\|\tilde{N}(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \int_0^t \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} d\tau \lesssim \|\tilde{N}_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \int_0^t (1 + \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\tilde{N}\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} d\tau.$$

Clearly, if the solution is small enough (which is ensured if the initial data is small) then, the last term of the right-hand side may be absorbed by the left-hand side, leading to Inequality (140). The above formal inequalities combined with a suitable contracting mapping argument (in the spirit of the one that is used e.g. for solving the incompressible Navier-Stokes equations, see details in [2, Chap. 5]), allow to conclude to the global existence of a solution to (125), fulfilling the desired properties.

To finish the proof of Theorem 4.3, we just have to compare \tilde{Z}_1 with \tilde{N} . To proceed, let us subtract (125) from (123). We get the following equation for $\delta\tilde{N} \triangleq \tilde{Z}_1 - \tilde{N}$:

$$\begin{aligned} \partial_\tau \delta\tilde{N} + \mathcal{A}\delta\tilde{N} &= S - Q_1(\tilde{Z}_1, \nabla^2 \delta\tilde{N}) - Q_1(\delta\tilde{N}, \nabla^2 \tilde{N}) - Q_2(\nabla \tilde{Z}_1, \nabla \delta\tilde{N}) - Q_2(\nabla \delta\tilde{N}, \nabla \tilde{N}) \\ &\quad - T_1(\delta\tilde{N}, \nabla \tilde{Z}_1, \nabla \tilde{Z}_1) - T_1(\tilde{N}, \nabla \delta\tilde{N}, \nabla \tilde{Z}_1) - T_1(\tilde{N}, \nabla \tilde{N}, \nabla \delta\tilde{N}) \\ &\quad - T_2(\delta\tilde{N}, \tilde{Z}_1, \nabla^2 \tilde{Z}_1) - T_2(\tilde{N}, \delta\tilde{N}, \nabla^2 \tilde{Z}_1) - T_2(\tilde{N}, \tilde{N}, \nabla^2 \delta\tilde{N}). \end{aligned}$$

Hence, by virtue of Proposition A.1, we have for all $t \geq 0$,

$$\begin{aligned} \|\delta\tilde{N}\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1}) \cap L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} &\lesssim \|\delta\tilde{N}(0)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|S\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|Q_1(\tilde{Z}_1, \nabla^2 \delta\tilde{N})\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \\ &\quad + \|Q_1(\delta\tilde{N}, \nabla^2 \tilde{N})\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|Q_2(\nabla \tilde{Z}_1, \nabla \delta\tilde{N})\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|Q_2(\nabla \delta\tilde{N}, \nabla \tilde{N})\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \\ &\quad + \|T_1(\delta\tilde{N}, \nabla \tilde{Z}_1, \nabla \tilde{Z}_1)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|T_1(\tilde{N}, \nabla \delta\tilde{N}, \nabla \tilde{Z}_1)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|T_1(\tilde{N}, \nabla \tilde{N}, \nabla \delta\tilde{N})\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \\ &\quad + \|T_2(\delta\tilde{N}, \tilde{Z}_1, \nabla^2 \tilde{Z}_1)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|T_2(\tilde{N}, \delta\tilde{N}, \nabla^2 \tilde{Z}_1)\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|T_2(\tilde{N}, \tilde{N}, \nabla^2 \delta\tilde{N})\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})}. \end{aligned}$$

So, using (69), the stability of $\dot{B}_{2,1}^{\frac{d}{2}}$ by product, (145), (138) and (140), we find that

$$\begin{aligned} \|\delta\tilde{N}\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1}) \cap L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} &\lesssim \|\delta\tilde{N}(0)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|\tilde{S}\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \\ &+ \left((1 + \|(\tilde{N}, \tilde{Z}_1)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}) \|\tilde{N}\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + \|\tilde{Z}_1\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \right) \|\nabla \delta\tilde{N}\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} \\ &+ \|(\tilde{N}, \tilde{Z}_1)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla^2 \tilde{Z}_1\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\delta\tilde{N}\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})} \\ &+ \left(\|\nabla^2 \tilde{N}\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} + \|\nabla \tilde{Z}_1\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^2 \right) \|\delta\tilde{N}\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \\ &\lesssim \|\delta\tilde{N}(0)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \alpha\varepsilon + (\alpha + \alpha^2) \|\delta\tilde{N}\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1}) \cap L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})}. \end{aligned}$$

Hence, as α is small enough, we get:

$$(146) \quad \|\delta\tilde{N}\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1}) \cap L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \lesssim \|\delta\tilde{N}(0)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \alpha\varepsilon \quad \text{for all } t \geq 0,$$

which completes the proof of the theorem. \square

We end this section with a few remarks. The first one is that, for small ε , it is natural to modify the definition in (121) so as to have a damped mode that is expressed in terms of \tilde{Z}_2 and \tilde{N} . If we set

$$(147) \quad \tilde{W} \triangleq \tilde{Z}_2 + L_2^{-1} \sum_{k=1}^d (\tilde{A}_{21}^k + \tilde{A}_{21}^k(\tilde{N})) \partial_k \tilde{N},$$

then we have

$$\tilde{W} - \check{W} = L_2^{-1} \sum_{k=1}^d (\tilde{A}_{21}^k(\tilde{N}) \partial_k \delta\tilde{N} + \tilde{A}_{21}^k(\delta\tilde{N}) \partial_k \tilde{Z}_1).$$

In order to bound the right-hand side, one can observe that

$$\begin{aligned} \|\tilde{A}_{21}^k(\tilde{N}) \partial_k \delta\tilde{N}\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} &\lesssim \|\tilde{N}\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \|\delta\tilde{N}\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1})}, \\ \|\tilde{A}_{21}^k(\delta\tilde{N}) \partial_k \tilde{Z}_1\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} &\lesssim \|\delta\tilde{N}\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \|\partial_k \tilde{Z}_1\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})}. \end{aligned}$$

Hence, taking advantage of Inequalities (140), (141) and (142), and of interpolation inequalities yields

$$(148) \quad \|\tilde{W} - \check{W}\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \leq C\alpha (\|\delta\tilde{N}(0)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \alpha\varepsilon),$$

which guarantees that \tilde{W} satisfies (121).

Note also that, since \tilde{Z}_1 is bounded in $\mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$ independently of ε , using (140) and (142), and interpolating, one obtains

$$\|\tilde{N} - \tilde{Z}_1\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-\beta})} \leq C\varepsilon^\beta, \quad \beta \in (0, 1).$$

Finally, observe that if we introduce the following rescaled solution of the limit system:

$$N^\varepsilon(t, x) \triangleq \tilde{N}^\varepsilon(\varepsilon t, x),$$

then combining (142) with the definition of \tilde{Z}_1^ε in (119) yields

$$Z_1^\varepsilon = N^\varepsilon + \mathcal{O}(\varepsilon) \quad \text{in } L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})$$

which is, indeed, a more accurate information than what we had in Theorem 4.1 or in (118). Similarly, putting (142) and (148) together yields the following expansion:

$$Z_2^\varepsilon(t, x) = -\varepsilon L_2^{-1} \sum_{k=1}^d (\bar{A}_{21}^k + \tilde{A}_{21}^k(\tilde{N}(\varepsilon t, x))) \partial_k \tilde{N}(\varepsilon t, x) + \mathcal{O}(\varepsilon) \quad \text{in } L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}).$$

APPENDIX A

The following classical result (see the proof in e.g. the Appendix of [10]) has been used a number of times in this text.

Lemma A.1. *Let $X : [0, T] \rightarrow \mathbb{R}_+$ be a continuous function such that X^2 is differentiable. Assume that there exists a constant $c \geq 0$ and a measurable function $A : [0, T] \rightarrow \mathbb{R}_+$ such that*

$$\frac{1}{2} \frac{d}{dt} X^2 + cX^2 \leq AX \quad \text{a.e. on } [0, T].$$

Then, for all $t \in [0, T]$, we have

$$X(t) + c \int_0^t X(\tau) d\tau \leq X_0 + \int_0^t A(\tau) d\tau.$$

We frequently took advantage of the fact that applying derivatives or, more generally, Fourier multipliers on spectrally localized functions is almost equivalent to multiplying by some constant depending only on the Fourier multiplier and on the spectral support.

This is illustrated by the classical Bernstein inequality that states (see e.g. [2, Chap. 2]) that for any $R > 0$ there exists a constant C such that for any $\lambda > 0$ and any function $u : \mathbb{R}^d \rightarrow \mathbb{R}$ with Fourier transform \hat{u} supported in the ball $B(0, R\lambda)$, we have

$$(149) \quad \|\partial^\alpha u\|_{L^q} \leq C^{1+|\alpha|} \lambda^{|\alpha|+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}, \quad \alpha \in \mathbb{N}^d, \quad 1 \leq p \leq q \leq \infty.$$

The reverse Bernstein inequality asserts that, under the stronger assumption that \hat{u} is supported in the annulus $\{x \in \mathbb{R}^d : r\lambda \leq |x| \leq R\lambda\}$ for some $0 < r < R$, then we have in addition,

$$(150) \quad \|u\|_{L^p} \leq C\lambda^{-1} \|\nabla u\|_{L^p}, \quad 1 \leq p \leq \infty.$$

A slight modification of the proof of (149) allows to extend the result to any smooth homogeneous multiplier : denoting by M a smooth function on $\mathbb{R}^d \setminus \{0\}$ with homogeneity γ , there exists a constant C such that for any $\lambda > 0$ and any function $u : \mathbb{R}^d \rightarrow \mathbb{R}$ with Fourier transform \hat{u} supported in the annulus $\{x \in \mathbb{R}^d : r\lambda \leq |x| \leq R\lambda\}$, we have

$$(151) \quad \|M(D)u\|_{L^q} \leq C\lambda^{\gamma+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}, \quad \alpha \in \mathbb{N}^d, \quad 1 \leq p \leq q \leq \infty.$$

In the last section, in order to study the convergence to the limit system, we used maximal regularity estimates in Besov spaces with last index 1 for parabolic system. These estimates are well known for the heat equation (see e.g. [2, Chap. 2]). Below, we extend them to semi-groups generated by strictly elliptic homogeneous multipliers in the following meaning: we consider functions $\mathcal{A} \in \mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\}; \mathcal{M}_n(\mathbb{C}))$ homogeneous of degree γ , such that the matrix $\mathcal{A}(\xi)$ is Hermitian and satisfies for some $c > 0$:

$$(152) \quad (\mathcal{A}(\xi)z \cdot z) \geq c|\xi|^\gamma |z|^2, \quad \xi \in \mathbb{R}^d \setminus \{0\}, \quad z \in \mathbb{C}^n.$$

Proposition A.1. *Let $u \in \mathcal{C}(\mathbb{R}_+; \mathcal{S}')$ satisfy*

$$(153) \quad \begin{cases} \partial_t z + \mathcal{A}(D)z = f & \text{on } \mathbb{R}_+ \times \mathbb{R}^d, \\ z|_{t=0} = z_0 & \text{on } \mathbb{R}^d. \end{cases}$$

Then, for any $p \in [1, \infty]$ and $s \in \mathbb{R}$ the following inequality holds true for all $t > 0$:

$$(154) \quad \|z(t)\|_{\dot{B}_{p,1}^s} + \int_0^t \|z\|_{\dot{B}_{p,1}^{s+\gamma}} d\tau \leq C \left(\|z_0\|_{\dot{B}_{p,1}^s} + \int_0^t \|f\|_{\dot{B}_{p,1}^s} d\tau \right).$$

Proof. If z satisfies (153) then for any $j \in \mathbb{Z}$, we have

$$\partial_t z_j + \mathcal{A}(D)z_j = f_j \quad \text{with} \quad z_j \triangleq \dot{\Delta}_j z \quad \text{and} \quad f_j \triangleq \dot{\Delta}_j f.$$

Hence, according to Duhamel's formula,

$$(155) \quad z_j(t) = e^{-t\mathcal{A}(D)} z_{0,j} + \int_0^t e^{-(t-\tau)\mathcal{A}(D)} f_j(\tau) d\tau.$$

Let us provisionally admit the following lemma:

Lemma A.2. *There exist two constants c_0 and C such that the following inequality holds for all $j \in \mathbb{Z}$, $t \geq 0$ and $p \in [1, \infty]$:*

$$(156) \quad \|\dot{\Delta}_j e^{t\mathcal{A}(D)} z\|_{L^p} \leq C e^{-c_0 2^{\gamma j} t} \|\dot{\Delta}_j z\|_{L^p}.$$

Then, plugging (156) in (155) yields for all $j \in \mathbb{Z}$,

$$(157) \quad \|z_j(t)\|_{L^p} \lesssim e^{-c_0 2^{\gamma j} t} \|z_{0,j}\|_{L^p} + \int_0^t e^{-c_0 2^{\gamma j} (t-\tau)} \|f_j(\tau)\|_{L^p} d\tau.$$

Hence, taking the supremum norm on $[0, t]$ (resp. integrating on $[0, t]$), we get for all $j \in \mathbb{Z}$,

$$\|z_j\|_{L^\infty(0,t;L^p)} + 2^{\gamma j} \|z_j\|_{L^1(0,t;L^p)} \lesssim \left(1 - e^{-c_0 2^{\gamma j} t}\right) \left(\|z_{0,j}\|_{L^p} + \|f_j\|_{L^1(0,t;L^p)}\right).$$

Just bounding the prefactor in the right-hand side by 1, multiplying the two sides by 2^{js} and summing up on \mathbb{Z} yields the claimed inequality. \square

Proof of Lemma A.2. Thanks to the homogeneity of \mathcal{A} , using a suitable change of variables reduces the proof to the case $j = 0$. Indeed, if we set $\zeta(x) \triangleq z(2^{-j}x)$, then we have $\dot{\Delta}_0 \zeta(2^j x) = \dot{\Delta}_j z(x)$ and

$$e^{-2^{\gamma j} \lambda \mathcal{A}(D)} \dot{\Delta}_0 \zeta(2^j x) = e^{-\lambda \mathcal{A}(D)} \dot{\Delta}_j z(x), \quad \lambda \geq 0.$$

Then, consider a function ϕ in $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$ with value 1 on a neighborhood of the support of φ and write that

$$\begin{aligned} e^{-t\mathcal{A}(D)} \dot{\Delta}_0 \zeta &= \mathcal{F}^{-1} \left(\phi e^{-\lambda \mathcal{A}(\cdot)} \widehat{\dot{\Delta}_0 \zeta} \right) \\ &= g_\lambda \star \dot{\Delta}_0 u \quad \text{with} \quad g_\lambda(x) \triangleq (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \phi(\xi) e^{-\lambda \mathcal{A}(\xi)} d\xi. \end{aligned}$$

If it is true that

$$(158) \quad \|g_\lambda\|_{L^1} \leq C e^{-c_0 \lambda}$$

then using that the convolution product maps $L^1 \star L^p$ to L^p implies that

$$\|e^{-\lambda \mathcal{A}(D)} \dot{\Delta}_0 \zeta\|_{L^p} \leq \|g_\lambda\|_{L^1} \|\dot{\Delta}_0 \zeta\|_{L^p} \leq C e^{-c_0 \lambda} \|\dot{\Delta}_0 \zeta\|_{L^p},$$

and we get (156) after reverting to the original variables.

In order to prove (158), it suffices to establish that

$$|g_\lambda(x)| \leq C(1 + |x|^2)^{-d} e^{-c_0\lambda}, \quad x \in \mathbb{R}^d, \quad \lambda > 0.$$

Now, integrating by parts, we get

$$(2\pi)^d g_\lambda(x) = (1 + |x|^2)^{-d} h_\lambda(x) \quad \text{with} \quad h_\lambda(x) \triangleq \int_{\mathbb{R}^d} e^{ix \cdot \xi} \phi(\xi) (\text{Id} - \Delta_\xi)^d (e^{-\lambda \mathcal{A}(\xi)}) d\xi.$$

Of course, the integral may be restricted to $\text{Supp } \phi$ which is a compact subset of $\mathbb{R}^d \setminus \{0\}$. Owing to (152), on this subset, there exists a positive constant c_0 such that all the real parts of the eigenvalues of $\mathcal{A}(\xi)$ are bounded from below by $2c_0$. Now, since the differential of the exponential map may be computed by the formula

$$D e^X \cdot H = \int_0^1 e^{(1-\tau)X} H e^{\tau X} d\tau, \quad H \in \mathcal{M}_n(\mathbb{R}),$$

the chain rule entails that

$$D_\xi \left(e^{-\lambda \mathcal{A}(\xi)} \right) \cdot H = -\lambda \int_0^1 e^{-\lambda(1-\tau)\mathcal{A}(\xi)} (D_\xi \mathcal{A}(\xi) \cdot H) e^{-\lambda\tau\mathcal{A}(\xi)} d\tau.$$

Hence, there exist two constants C and C' such that

$$\left| D_\xi \left(e^{-\lambda \mathcal{A}(\xi)} \right) \right| \leq C' \lambda e^{-2c_0\lambda} \leq C e^{-c_0\lambda}, \quad \lambda > 0, \quad \xi \in \text{Supp } \phi.$$

By induction, one can get similar estimates for higher order derivatives of $\xi \mapsto e^{-\lambda \mathcal{A}(\xi)}$, which eventually yields

$$|h_\lambda(x)| \leq C e^{-c_0\lambda}, \quad x \in \mathbb{R}^d, \quad \lambda > 0,$$

and completes the proof. \square

Remark A.1. *In the case $p = 2$ one can work out a shorter proof, based on the Fourier-Plancherel theorem. However, it is interesting to point out that the very same result holds for any value of p in $[1, \infty]$ including 1 and ∞ , and with a constant independent of p .*

The following lemma ensures that in the setting of System (57), if both Condition (SK) and $A_{11}^k(\bar{V}) = 0$ for all $k \in \{1, \dots, d\}$ are satisfied, then the second order differential operator \mathcal{A} defined in (122) is indeed strictly elliptic in the sense of Proposition A.1, with $\gamma = 2$.

Lemma A.3. *Consider two $n \times n$ Hermitian matrices A and B such that*

$$(159) \quad A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & B_{22} \end{pmatrix}$$

with $A_{12} \in \mathcal{M}_{n_1, n_2}(\mathbb{C})$, $A_{21} \in \mathcal{M}_{n_2, n_1}(\mathbb{C})$, $A_{22} \in \mathcal{M}_{n_2, n_2}(\mathbb{C})$ and $B_{22} \in \mathcal{M}_{n_2, n_2}(\mathbb{C})$. Suppose also that B_{22} is positive. Then, B_{22} is invertible and the following two properties are equivalent:

- (1) *The matrix $A_{12} B_{22}^{-1} A_{21}$ is a $n_1 \times n_1$ positive matrix.*
- (2) *Condition (SK) holds true (that is, the four equivalent conditions of Lemma 1.1 are satisfied).*

Proof. The invertibility of B_{22} being obvious, let us first assume that $A_{12}B_{22}^{-1}A_{21}$ is positive. Then, the rank of A_{21} must be equal to n_1 and so does the rank of $B_{22}A_{21}$. Now, we observe that

$$BA = \begin{pmatrix} 0 & 0 \\ B_{22}A_{21} & B_{22}A_{22} \end{pmatrix}.$$

Hence, the rank of $\begin{pmatrix} B \\ BA \end{pmatrix}$ is equal to $n_1 + n_2 = n$, and Condition (SK) is thus satisfied.

Conversely, if A has the special structure (159) then an easy induction reveals that the bottom left block of any positive power k of A ends with A_{21} . The same property clearly holds for BA^k that thus looks like

$$BA^k = \begin{pmatrix} 0 & 0 \\ B_{22}C_kA_{21} & D_k \end{pmatrix} \quad \text{for some } C_k, D_k \in \mathcal{M}_{n_2}(\mathbb{C}).$$

Now, since B_{22} is invertible, we have for all $k \in \mathbb{N}$,

$$\text{rank}(B_{22}C_kA_{21}) \leq \text{rank}(A_{21}) = \text{rank}(B_{22}A_{21}).$$

As the block at the bottom left of BA is equal to $B_{22}A_{21}$, one can conclude that, under assumption (159) we automatically have

$$\text{rank} \begin{pmatrix} B \\ BA \end{pmatrix} = \text{rank} \begin{pmatrix} B \\ BA \\ \dots \\ BA^{n-1} \end{pmatrix}.$$

Hence, if we assume in addition that Condition (SK) is satisfied, then we must have $\text{rank}(B_{22}A_{21}) = n_1$, and thus $\text{rank}(A_{21}) = n_1$, too. Now, since ${}^t\bar{A}_{12} = A_{21}$, we have for all $z \in \mathbb{C}^{n_1}$,

$$A_{12}B_{22}^{-1}A_{21}z \cdot z = B_{22}^{-1}A_{21}z \cdot A_{21}z.$$

As B_{22}^{-1} is positive, the right-hand side is nonnegative and vanishes if and only if $A_{21}z = 0$ and thus if and only if $z = 0$ since $\text{rank}(A_{21}) = n_1$. Hence $A_{12}B_{22}^{-1}A_{21}$ is positive, which completes the proof. \square

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